CSIR-NET – MATHEMATICAL SCIENCES
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PART - B (Mathematical Sciences)

1. For $t \in \mathbb{R}$, define $M(t) = \begin{pmatrix} 1 & t & 0 \\ 1 & 1 & t^2 \\ 0 & 1 & 1 \end{pmatrix}$.

Then which of the following statements is TRUE ?

(a) det $M(t)$ is a polynomial function of degree 3 in t.

(b) det $M(t) = 0$ for all $t \in \mathbb{R}$.

(c) det $M(t)$ is zero for infinitely many $t \in \mathbb{R}$.

(d) det $M(t)$ is zero for exactly two $t \in \mathbb{R}$.

2. Let $\leq$ be the usual order on the field $\mathbb{R}$ of real numbers. Define an order $\leq$ on $\mathbb{R}^2$ by $(a,b) \leq (c,d)$ if $(a < c)$, or $(a = c$ and $b \leq d)$. Consider the subset $E = \left\{\left(\frac{1}{n}, 1-\frac{1}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\}$. With respect to $\leq$ which of the following statements is TRUE ?

(a) $\inf(E) = (0,1)$ and $\sup(E) = (1,0)$

(b) $\inf(E)$ does not exist but $\sup(E) = (1,0)$

(c) $\inf(E) = (0,1)$ but $\sup(E)$ does not exist

(d) Both $\inf(E)$ and $\sup(E)$ do not exist

3. For a quadratic form in 3 variables over $\mathbb{R}$, let 'r' be the rank and 's' be the signature. The number of possible pairs (r, s) is :

(a) 13 (b) 9 (c) 10 (d) 16

4. Let $E = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$. For each $m \in \mathbb{N}$ define $f_m : E \to \mathbb{R}$ by:

$$f_m(x) = \begin{cases} \cos(mx) & \text{if} \quad x \geq \frac{1}{m} \\ 0 & \text{if} \quad \frac{1}{m+10} < x < \frac{1}{m} \\ x & \text{if} \quad x \leq \frac{1}{m+10} \end{cases}$$

Then which of the following statements is true ?

(a) No subsequence of $(f_m)_{m\geq 1}$ converges at every point of E.

(b) Every subsequence of $(f_m)_{m\geq 1}$ converges at every point of E.

(c) There exist infinitely many subsequence of $(f_m)_{m\geq 1}$ which converge at every point of E.

(d) There exists a subsequence of $(f_m)_{m\geq 1}$ which converges to 0 at every point of E.

5. Let $M_4(\mathbb{R})$ be the space of all (4×4) matrices over $\mathbb{R}$. Let

$$W = \left\{ (a_{ij}) \in M_4(\mathbb{R}) \,\middle|\, \sum_{i+j=k} a_{ij} = 0, \text{ for } k = 2, 3, 4, 5, 6, 7, 8 \right\}$$

Then $\dim(W)$ is :

(a) 7 (b) 8 (c) 9 (d) 10

6. Let V be a vector space of dimension 3 over $\mathbb{R}$. Let $T : V \to V$ be a linear transformation, given by the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -4 & 3 \\ -2 & 5 & -3 \end{pmatrix}$ with respect to an ordered basis $\{v_1, v_2, v_3\}$ of V. Then which of the following statements is TRUE ?

(a) $T(v_3) = 0$ (b) $T(v_1 + v_2) = 0$

(c) $T(v_1 + v_2 + v_3) = 0$ (d) $T(v_1 + v_3) = T(v_2)$

7. Let $X \subset \mathbb{R}$ be an infinite countable bounded subset of $\mathbb{R}$. Which of the following statements is TRUE ?

(a) X cannot be compact (b) X contains an interior point

(c) X may be closed (d) Closure of X is countable

8. Which of the following sets is countable ?

(a) The set of all functions from $\mathbb{Q}$ to $\mathbb{Q}$.

(b) The set of all functions from $\mathbb{Q}$ to $\{0, 1\}$.

(c) The set of all functions from $\mathbb{Q}$ to $\{0, 1\}$ which vanish outside a finite set.

(d) The set of all subsets of $\mathbb{N}$.

9. Let $(x_n)_{n\geq 1}$ be a sequence of non-negative real numbers. Then which of the following is true ?

(a) $\liminf x_n = 0 \Rightarrow \lim x_n^2 = 0$ (b) $\limsup x_n = 0 \Rightarrow \lim x_n^2 = 0$

(c) $\liminf x_n = 0 \Rightarrow (x_n)_{n\geq 1}$ is bounded (d) $\liminf x_n^2 > 4 \Rightarrow \limsup x_n > 4$

10. Let $C[0,1]$ be the space of continuous real valued functions on $[0,1]$. Define

$$\langle f, g \rangle = \int_0^1 f(t)(g(t))^2 \, dt \text{ for all } f, g \in C[0,1]$$

Then which of the following statements is true ?

(a) $\langle\ ,\ \rangle$ is an inner product on $C[0,1]$.

(b) $\langle\ ,\ \rangle$ is a bilinear form on $C[0,1]$ but is not an inner product on $C[0,1]$.

(c) $\langle\ ,\ \rangle$ is not a bilinear form on $C[0,1]$.

(d) $\langle f, f\rangle \geq 0$ for all $f \in C[0,1]$.

11. Let $A = \begin{pmatrix} 2 & 0 & 5 \\ 1 & 2 & 3 \\ -1 & 5 & 1 \end{pmatrix}$. The system of linear equation $AX = Y$ has a solution —

(a) only for $Y = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, x \in \mathbb{R}$

(b) only for $Y = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, y \in \mathbb{R}$

(c) only for $Y = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}, y, z \in \mathbb{R}$

(d) for all $Y \in \mathbb{R}^3$

12. What is the sum of the following series ?

$$\left(\frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 3}\right) + \left(\frac{1}{2^2\cdot 3^2} + \frac{1}{2^3\cdot 3^2}\right) + \cdots + \left(\frac{1}{2^a\cdot 3^a} + \frac{1}{2^{a+1}\cdot 3^a}\right) + \cdots$$

(a) $\frac{3}{8}$ (b) $\frac{3}{10}$ (c) $\frac{3}{14}$ (d) $\frac{3}{16}$

13. A permutation σ of $[n] = \{1, 2, ..., n\}$ is called irreducible, if the restriction $\sigma|_{[k]}$ is not a permutation of $[k]$ for any $1 \leq k < n$. Let a_n be the number of irreducible permutations of $[n]$. Then $a_1 = 1$, $a_2 = 1$ and $a_3 = 3$. The value of a_4 is :

(a) 12 (b) 13 (c) 14 (d) 15

14. Let X be an infinite set. Consider the topology τ on X whose non-empty open sets are complements of finite sets. Then which of the following statements is true ?

(a) X is disconnected

(b) X is compact

(c) No sequence in X converges in X

(d) Every sequence in X converges to a unique point in X

15. Let $T : \mathbb{C} \to \mathbb{M}_2(\mathbb{R})$ be the map given by:

$$T(z) = T(x+iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Then which of the following statements is false ?

(a) $T(z_1 z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \mathbb{C}$

(b) $T(z)$ is singular if and only if $z = 0$

(c) There does not exist non-zero $A \in \mathbb{M}_2(\mathbb{R})$ such that the trace of $T(z)A$ is zero for all $z \in \mathbb{C}$

(d) $T(z_1 + z_2) = T(z_1) + T(z_2)$ for all $z_1, z_2 \in \mathbb{C}$

16. Let S_5 be the symmetric group on five symbols. Then which of the following statements is false ?

(a) S_5 contains a cyclic subgroup of order 6.

(b) S_5 contains a non-abelian subgroup of order 8.

(c) S_5 does not contain a subgroup isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$.

(d) S_5 does not contain a subgroup of order 7.

17. Consider the polynomial $f(z) = z^2 + az + p^{11}$, where $a \in \mathbb{Z} \setminus \{0\}$ and $p \geq 13$ is a prime. Suppose that $a^2 \leq 4p^{11}$. Which of the following statements is true ?

(a) f has a zero on the imaginary axis.

(b) f has a zero for which the real and imaginary parts are equal.

(c) f has distinct roots.

(d) f has exactly one real root.

18. Let G be a group of order p^n, p a prime number and $n > 1$. Then which of the following is true ?

(a) Centre of G has at least two elements.

(b) G is always an Abelian group.

(c) G has exactly two normal subgroups (i.e., G is a simple group).

(d) If H is any other group of order p^n, then G is isomorphic to H.

19. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then which of the following statements is true ?

(a) No such f exists

(b) Such an f is not unique

(c) $f(z) = z^2$ for all $z \in \mathbb{C}$

(d) f need not be a polynomial function

20. For $z \in \mathbb{C}$, let $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Then which of the following statements is false ?

(a) $f(z)$ is continuous everywhere.

(b) $f(z)$ is not analytic in any open neighbourhood of zero.

(c) $z f(z)$ satisfies the Cauchy-Riemann equations at zero.

(d) $f(z)$ is analytic in some open subset of $\mathbb{C}$.

21. Let ϕ be the solution of $\phi(x) = 1 - 2x - 4x^2 + \int_0^x \left[3 + 6(x-t) - 4(x-t)^2\right] \phi(t)\, dt$. Then $\phi(1)$ is equal to:

(a) e^{-1}

(b) e^{-2}

(c) e

(d) e^2

22. Let $x = \xi$ be a solution of $x^4 - 3x^2 + x - 10 = 0$. The rate of convergence for the iterative method $x_{n+1} = 10 - x_n^4 + 3x_n^2$ is equal to:

(a) 1 (b) 2 (c) 3 (d) 4

23. For the following system of ordinary differential equations:

$$\frac{dx}{dt} = x(3 - 2x - 2y), \ \& \ \frac{dy}{dt} = y(2 - 2x - y)$$

the critical point $(0, 2)$ is:

(a) a stable spiral (b) an unstable spiral (c) a stable node (d) an unstable node

24. Let $y = \phi(x)$ be the extremizing function for the functional $I(y) = \int_0^1 y^2 \left(\frac{dy}{dx}\right)^2 dx$, subject to $y(0) = 0$, $y(1) = 1$. Then $\phi\left(\frac{1}{4}\right)$ is equal to:

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{12}$

25. Consider the system of ordinary differential equations:

$$\frac{dx}{dt} = 4x^3y^2 - x^5y^4, \ \& \ \frac{dy}{dt} = x^4y^5 + 2x^2y^3$$

Then for this system there exists

(a) a closed path in $\left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 5\right\}$

(b) a closed path in $\left\{(x, y) \in \mathbb{R}^2 \mid 5 < x^2 + y^2 \le 10\right\}$

(c) a closed path in $\left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 10\right\}$

(d) no closed path in $\mathbb{R}^2$

26. Let $u(x, y)$ be the solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 64$ in the unit disc $\left\{(x, y) \mid x^2 + y^2 < 1\right\}$ and such that u vanishes on the boundary of the disc. Then $u\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ is equal to:

(a) 7 (b) 16 (c) -7 (d) -16

27. Consider a mass-less infinite straight wire with one end fixed at *O*. Assume that the wire is rotating in a plane about the point *O* with constant angular velocity ω. Consider a bead of mass *m* sliding along the wire in the absence of external forces. Let $r(t)$ denote the distance of the bead from *O* at time $t \ge 0$, and $\frac{dr}{dt}(0) = 0$. Then which of the following statements is true ?

(a) $\exists M > 0, \alpha > 0$ such that $r(t) > Me^{\alpha t}, t > 0$

(b) $r(t) \to 0$ as $t \to \infty$

(c) $r(t)$ is a constant function

(d) $\frac{dr(t)}{dt}$ changes its sign for some $t > 0$

28. The Cauchy problem $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ and $x_0(s) = \cos(s),\ y_0(s) = \sin(s),\ z_0(s) = 1,\ \ s > 0$ has

(a) a unique solution
(b) no solution
(c) more than one but finite number of solutions
(d) infinitely many solutions

29. In a 2^3 factorial design, the treatment combinations of three treatments A, B and C are allotted to 2 blocks of 4 plots each. Suppose the key block is as follows:

Key block (1), a, bc, abc

Then the confounded treatment combination is :

(a) AB (b) AC (c) BC (d) ABC

30. Suppose data $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are generated as follows: $Y_1, Y_2, ..., Y_n \sim$ i.i.d. Bernoulli $\left(\frac{1}{2}\right)$ and $X_i \mid Y_i = y \sim$ Uniform $(0, y+1)$. Define $h(x) = \begin{cases} 1 & \text{if} \quad x \geq 1 \\ 0 & \text{otherwise} \end{cases}$

Then, which of the following is a correct linear regression model for $m(x) = E(Y_i \mid X_i = x)$, in the sense that the true $m(x)$ is obtained for some values of the parameters $\alpha_0, \alpha_1, \alpha_2$ for all x ?

(a) $m(x) = \alpha_0 + \alpha_1 x$
(b) $m(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$
(c) $m(x) = \alpha_0 + \alpha_1 x + \alpha_2 x\, h(x)$
(d) $m(x) = \alpha_0 + \alpha_1 x + \alpha_2\, h(x)$

31. Let X_1 and X_2 be a random sample of size 2 from uniform $[0, \theta]$ distribution, $\theta > 0$.

Define $M = \max\{X_1, X_2\}$. What is the confidence coefficient of the confidence interval $\left[\frac{3}{7}M, 2M\right]$ for θ ?

(a) 0.6285 (b) 0.75 (c) 0.8333 (d) 0.95

32. Let X be a real valued random variable such that $E[e^X] < \infty$ and $E[e^X] = e^{E[X]}$. Then which of the following is correct ?

(a) $P(X \geq a) \geq e^{E[X]-a}$ for all $a \in \mathbb{R}$
(b) $E[X^3] = (E[X])^3$
(c) $\text{Var}(X) \neq 0$
(d) $X \geq 0$ almost surely

33. Suppose $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}$ are i.i.d. observations from the Uniform distribution on the unit square [0, 1] × [0, 1]. What is the probability that the rank correlation between the X_i and the Y_i values is 1 ?

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

34. Let X and Y be independent Exponential random variable with means $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively with $\lambda \neq \mu$. Let $f_z(z)$ denote the density function of $Z = X + Y$. Then for $z > 0$.

(a) $f_z(z) = (\lambda + \mu)e^{-(\lambda+\mu)z}$

(b) $f_z(z) = \frac{\lambda\mu}{\lambda+\mu} e^{\frac{-\lambda\mu}{\lambda+\mu}z}$

(c) $f_z(z) = \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu z} - e^{-\lambda z})$

(d) $f_z(z) = \begin{cases} \frac{\lambda\mu}{\lambda-\mu} e^{\frac{-\lambda\mu}{\lambda-\mu}z} & \text{if } \lambda > \mu \\ \frac{\lambda\mu}{\mu-\lambda} e^{\frac{-\lambda\mu}{\mu-\lambda}z} & \text{if } \mu > \lambda \end{cases}$

35. Let $X_1, X_2, ..., X_n$ $(n \geq 2)$ be a random sample from a distribution with probability density function $f_\theta, \theta > 0$, unknown, where

$$f_\theta(x) = \begin{cases} \frac{2(\theta - x)}{\theta^2}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

Let $\overline{X}_n$ be the sample mean and $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$.
Then which of the following statements is ***correct*** ?

(a) $X_{(n)}$ is sufficient for θ.

(b) $X_{(n)}$ is unbiased for θ.

(c) $3\overline{X}_n$ is unbiased for θ.

(d) $3\overline{X}_n$ is sufficient for θ.

36. To draw a sample of size $n(\geq 5)$ using a without replacement scheme from a finite population $\{U_1, U_2, ..., U_N\}$ of size N, the first unit is chosen using $PPS(p_1, p_2, ... p_N)$ scheme and the remaining $(n-1)$ units are drawn using SRSWOR. Then the probability that U_2 is incident in the sample is :

(a) $\frac{N-n}{N-1}p_2 + \frac{1}{N-1}$

(b) $\frac{N-n}{N-1}p_2 + \frac{n-2}{N-1}$

(c) $\frac{N-n}{N-1}p_2 + \frac{N-n}{N-1}$

(d) $\frac{N-n}{N-1}p_2 + \frac{n-1}{N-1}$

37. Subject to the condition, $0 \leq x \leq 10,\ 0 \leq y \leq 5$, the minimum value of the function $4x - 5y + 10$ is :

(a) 10 (b) 0 (c) –25 (d) –15

38. There are three urns U_1, U_2, U_3, each with balls of two colours. U_1 contains 2 white balls and 3 black balls, U_2 contains 3 white balls and 2 black balls and U_3 contains 5 white balls and 5 black balls. An urn is chosen at random and a ball is drawn from that urn at random. What is the probability that U_2 was chosen given that the ball picked is black in colour ?

(a) $\frac{1}{3}$ (b) $\frac{4}{15}$ (c) $\frac{2}{15}$ (d) $\frac{1}{6}$

39. Let $(X_1, Y_1), (X_2, Y_2) .... (X_n, Y_n)$, $n \geq 5$, be a random sample from a Bivariate Normal $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ distribution with all parameters unknown. For testing $H_0 : \rho = 0$ against $H_1 : \rho \neq 0$ if you use the usual t-test and your observed sample correlation coefficient is 0, then what is the p-value ?

(a) 0 (b) 0.05 (c) 0.5 (d) 1

40. Let $\{X_n : n \geq 0\}$ be a two state Markov chain with state space S. $\{0, 1\}$ and transition matrix $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$.

Assuming $X_0 = 0$, the expected return time to 0 is :

(a) $\frac{5}{2}$ (b) $\frac{9}{4}$ (c) $\frac{3}{2}$ (d) 3

PART - C (Mathematical Sciences)

1. For each natural number $n \geq 1$, let $a_n = \frac{n}{10^{\lceil \log_{10} n \rceil}}$, where $[x]$ = smallest integer greater than or equal to x. Which of the following statements are true ?

(a) $\liminf_{n\to\infty} a_n = 0$

(b) $\liminf_{n\to\infty} a_n$ does not exist

(c) $\liminf_{n\to\infty} a_n = 0.15$

(d) $\limsup_{n\to\infty} a_n = 1$

2. Let n be a fixed natural number. Then the series $\sum_{m \geq n} \frac{(-1)^m}{m}$ is :

(a) Absolutely convergent

(b) Divergent

(c) Absolutely convergent if $n > 100$

(d) Convergent

3. Let $N \geq 5$ be an integer. Then which of the following statements are ***true*** ?

(a) $\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N$ (b) $\sum_{n=1}^{N} \frac{1}{n} < 1 + \log N$ (c) $\sum_{n=1}^{N} \frac{1}{n} \leq \log N$ (d) $\sum_{n=1}^{N} \frac{1}{n} \geq \log N$

4. Let $A \in M_3(\mathbb{R})$ and let $X = \{C \in GL_3(\mathbb{R}) \mid CAC^{-1} \text{ is triangular}\}$. Then

(a) $X \neq \phi$

(b) If $X = \phi$, then A is not diagonalizable over $\mathbb{C}$.

(c) If $X = \phi$, then A is diagonalizable over $\mathbb{C}$.

(d) If $X = \phi$, then A has no real eigenvalue.

5. Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^∞ - function. Suppose that for every $x \in U$, the derivative at x, $d f_x$, is non singular. Then which of the following statements are true ?

(a) If $V \subset U$ is open then $f(V)$ is open in $\mathbb{R}^n$.

(b) $f : U \to f(U)$ is a homeomorphism.

(c) f is one-one.

(d) If $V \subset U$ is closed, then $f(V)$ is closed in $\mathbb{R}^n$.

6. Let X be a finite dimensional inner product space over $\mathbb{C}$. Let $T : X \to X$ be any linear transformation. Then which of the following statements are true ?

(a) T is unitary $\Rightarrow T$ is self adjoint.

(b) T is self adjoint $\Rightarrow T$ is normal.

(c) T is unitary $\Rightarrow T$ is normal.

(d) T is normal $\Rightarrow T$ is unitary.

7. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation, $n \geq 2$. Suppose 1 is the only eigenvalue of T.

(a) $T^k \neq I$ for any $k \in \mathbb{N}$

(b) $(T - I)^{n-1} = 0$

(c) $(T - I)^n = 0$

(d) $(T - I)^{n+1} = 0$

8. Let $\{a_n\}_{n \geq 1}$ be a bounded sequence of real numbers. Then —

(a) Every subsequence of $\{a_n\}_{n \geq 1}$ is convergent.

(b) There is exactly one subsequence of $\{a_n\}_{n \geq 1}$ which is convergent.

(c) There are infinitely many subsequence of $\{a_n\}_{n \geq 1}$ which are convergent.

(d) There is a subsequence of $\{a_n\}_{n \geq 1}$ which is convergent.

9. Let (X, d) be a compact metric space. Let $T : X \to X$ be a continuous function satisfying $\inf\limits_{n \in \mathbb{N}} d(T^n(x), T^n(y)) \neq 0$ for every $x, y \in X$ with $x \neq y$. Then which of the following statements are true ?

(a) T is a one-one function.

(b) T is not a one-one function.

(c) Image of T is closed in X.

(d) If X is finite, then T is onto.

10. Which of the following statements are true ?

(a) Any two quadratic forms of same rank in n-variables over $\mathbb{R}$ are isomorphic.

(b) Any two quadratic forms of same rank in n-variables over $\mathbb{C}$ are isomorphic.

(c) Any two quadratic forms in n-variables are isomorphic over $\mathbb{C}$.

(d) A quadratic form in 4 variables may be isomorphic to a quadratic form in 10 variables.

11. Let $f : [0, 1] \to \mathbb{R}$ be a monotomic function with $f\left(\frac{1}{4}\right) f\left(\frac{3}{4}\right) < 0$. Suppose $\sup\{x \in [0, 1] : f(x) < 0\} = \alpha$.

Which of the following statements are correct ?

(a) $f(\alpha) < 0$

(b) If f is increasing, then $f(\alpha) \leq 0$

(c) If f is continuous and $\frac{1}{4} < \alpha < \frac{3}{4}$, then $f(\alpha) = 0$

(d) If f is decreasing, then $f(\alpha) < 0$

12. Let $p(x)$ be a polynomial function in one variable of odd degree and g be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then which of the following statements are true ?

(a) $\exists$ a point $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(b) If g is a polynomial function then there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(c) If g is a bounded function there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(d) There is a unique point $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

13. Let $n \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Suppose $A_n(\alpha, \beta) = [a_{ij}]$ is an $n \times n$ matrix such that $a_{ii} = \alpha$ and $a_{ij} = \beta$ for $i \neq j, 1 \leq i, j \leq n$. Let D_n be the determinant of $A_n(\alpha, \beta)$. Which of the following statements are true ?

(a) $D_n = (\alpha - \beta) D_{n+1} + \beta$ for $n \geq 2$

(b) $\dfrac{D_n}{(\alpha-\beta)^{n-1}} = \dfrac{D_{n-1}}{(\alpha-\beta)^{n-2}} + \beta$ for $n \geq 2$

(c) $D_n = (\alpha + (n-1)\beta)^{n-1}(\alpha - \beta)$ for $n \geq 2$

(d) $D_n = (\alpha + (n-1)\beta)(\alpha - \beta)^{n-1}$ for $n \geq 2$

14. Let $f(x)$ be a real polynomial of degree 4. Suppose $f(-1) = 0,\ f(0) = 0$. $f(1) = 1$ and $f^{(1)}(0) = 0$, where $f^{(k)}(a)$ is the value of k^{th} derivative of $f(x)$ at $x = a$. Which of the following statements are true ?

(a) There exists $a \in (-1, 1)$ such that $f^{(3)}(a) \geq 3$.

(b) $f^{(3)}(a) \geq 3$ for all $a \in (-1, 1)$.

(c) $0 < f^{(3)}(0) \geq 2$

(d) $f^{3}(0) \geq 3$

15. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation with characteristic polynomial $(x-2)^4$ and minimal polynomial $(x-2)^2$. Jordan canonical form of T can be :

(a) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

16. Let $L^2([-\pi, \pi])$ be the metric space of Lebesgue square integrable functions on $[-\pi, \pi]$ with a metric d given by:

$$d(f, g) = \left[\int_{-\pi}^{\pi} \left(f(x) - g(x) \right)^2 dx \right]^{1/2} \text{ for } f, g \in L^2\left([-\pi, \pi]\right)$$

Consider the subset, $S = \left\{ \sin(2^n x) : n \in \mathbb{N} \right\}$ of $L^2\left([-\pi, \pi]\right)$. Which of the following statements are true ?

(a) S is bounded (b) S is closed (c) S is compact (d) S is non-compact

17. Let $f:[0,1]^2 \to \mathbb{R}$ be a function defined by:

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if} \quad \text{either } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{if} \quad x = y = 0 \end{cases}$$

Then which of the following statements are true ?

(a) f is continuous at (0, 0)

(b) f is a bounded function

(c) $\int_0^1 \int_0^1 f(x, y)\, dx\, dy$ exists

(d) f is continuous at (1, 0)

18. Which of the following statements regarding quadratic forms in 3 variables are true ?

(a) Any two quadratic forms of rank 3 are isomorphic over $\mathbb{R}$.

(b) Any two quadratic forms of rank 3 are isomorphic over $\mathbb{C}$.

(c) There are exactly three non-zero quadratic forms of rank ≤ 3 upto isomorphism.

(d) There are exactly three non-zero quadratic forms of rank 2 upto isomorphism over $\mathbb{R}$ and $\mathbb{C}$.

19. Let $C[0,1]$ be the ring of all real valued continuous function on [0, 1].

Let $A = \{ f \in C[0,1] : f(1/4) = f(3/4) = 0 \}$. Then which of the following statements are true ?

(a) A is an ideal in $C[0,1]$ but is not a prime ideal in $C[0,1]$.

(b) A is a prime ideal in $C[0,1]$.

(c) A is a maximal ideal in $C[0,1]$.

(d) A is a prime ideal in $C[0,1]$, but is not a maximal ideal in $C[0,1]$.

20. Which of the following statements are true ?

(a) There exist three mutually disjoint subsets of $\mathbb{R}$, each of which is countable and dense in $\mathbb{R}$.

(b) For each $n \in \mathbb{N}$, there exist n mutually disjoint subsets of $\mathbb{R}$, each of which is countable and dense in $\mathbb{R}$.

(c) There exist countably infinite number of mutually disjoint subsets of $\mathbb{R}$, each of which is countable and dense in $\mathbb{R}$.

(d) There exist uncountable number of mutually disjoint subsets of $\mathbb{R}$, each of which is countable and dense in $\mathbb{R}$.

21. Consider the power series: $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n)!}$ Which of the following are true ?

(a) Radius of convergence of $f(z)$ is infinite. (b) The set $\{ f(x) : x \in \mathbb{R} \}$ is bounded.

(c) The set $\{f(x): -1 < x < 1\}$ is bounded. (d) $f(z)$ has infinitely many zeroes.

22. Let $F[X]$ be the polynomial ring in one variable over a field F. Then which of the following statements are true?

(a) $F[X]$ is a *UFD*
(b) $F[X]$ is a *PID*
(c) $F[X]$ is a Euclidean function
(d) $F[X]$ is a *PID* but is not an Euclidean domain

23. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n. Then which of the following are true ?

(a) If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$.
(b) If $f(x)$ is irreducible in $\mathbb{Q}[x]$, then it is irreducible in $\mathbb{Z}[x]$.
(c) If $f(x)$ is reducible in $\mathbb{Z}[x]$, then it has a real root.
(d) If $f(x)$ has a real root, then it is reducible in $\mathbb{Z}[x]$.

24. Let $f: \mathbb{C} \to \mathbb{C}$ be an analytic function. For $z_0 \in \mathbb{C}$, which of the following statements are true ?

(a) f can take the value z_0 at finitely many points in $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.
(b) $f(1/n) = z_0$ for all $n \in \mathbb{N} \Rightarrow f$ is the constant function z_0.
(c) $f(n) = z_0$ for all $n \in \mathbb{N} \Rightarrow f$ is the constant function z_0.
(d) $f(r) = z_0$ for all $r \in \mathbb{Q} \cap [1, 2] \Rightarrow f$ is the constant function z_0.

25. Let I be an ideal of $\mathbb{Z}$. Then which of the following statements are true ?

(a) I is a principal ideal.
(b) I is a prime ideal of $\mathbb{Z}$.
(c) If I is a prime ideal of $\mathbb{Z}$, then I is a maximal ideal in $\mathbb{Z}$.
(d) If I is a maximal ideal in $\mathbb{Z}$, then I is a prime ideal in $\mathbb{Z}$.

26. Let U be an open subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$ be an analytic function. Then which of the following are true?

(a) If f is one-one, then $f(U)$ is open in $\mathbb{C}$.
(b) If f is onto, then $U = \mathbb{C}$.
(c) If f is onto, then f is one-one.
(d) If $f(U)$ is closed in $\mathbb{C}$, then $f(U)$ is connected.

27. Consider $[n] = \{1, 2, ..., n\}$ with the discrete topology and let $X = \prod_{n \geq 1} [n]$ be the product space with the product topology. For $x = (a_1, a_2, ...) \in X$, define $T(x) = (1, a_1, a_2, ...)$. Then which of the following statements are true ?

(a) Let $x_n \in X$ for $n = 1, 2, 3, ...$ be a sequence in X. Then it is convergent.
(b) X is a compact, Hausdorff space
(c) The map $T: X \to X$ is continuous.
(d) The map $T: X \to X$ has a unique fixed point.

28. Let F be a field. Then which of the following statements are true ?

(a) All extensions of degree 2 of F are isomorphic as fields.

(b) All finite extensions of F of same degree are isomorphic as fields if $char(F) > 0$.

(c) All finite extensions of F of same degree are isomorphic as fields if F is finite.

(d) All finite normal extensions of F are isomorphic as field if $char(F) = 0$.

29. Let $U \subset \mathbb{C}$ be an open connected set and $f : U \to \mathbb{C}$ be a non-constant analytic function. Consider the following two sets:

$X = \{z \in U : f(z) = 0\}$

$Y = \{z \in U : f$ vanishes on an open neighbourhood of z in $U\}$

Then which of the following statements are true ?

(a) X is closed in U

(b) Y is closed in U

(c) X has empty interior

(d) Y is open in U

30. For a given integer k, which of the following statements are false ?

(a) If k (mod 72) is a unit in $\mathbb{Z}_{72}$, then k (mod 9) is a unit in $\mathbb{Z}_9$.

(b) If k (mod 72) is a unit in $\mathbb{Z}_{72}$, then k (mod 8) is a unit in $\mathbb{Z}_8$.

(c) If k (mod 8) is a unit in $\mathbb{Z}_8$, then k (mod 72) is a unit in $\mathbb{Z}_{72}$.

(d) If k (mod 9) is a unit in $\mathbb{Z}_9$, then k (mod 72) is a unit in $\mathbb{Z}_{72}$.

31. Consider the initial value problem $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1;\ 0 \le x \le 1$. Then which of the following statements are true ?

(a) There exists a unique solution in $\left[0, \frac{\pi}{4}\right]$

(b) Every solution is bounded in $\left[0, \frac{\pi}{4}\right]$

(c) The solution exhibits a singularity at some point in [0, 1]

(d) The solution becomes unbounded in some subinterval of $\left[\frac{\pi}{4}, 1\right]$

32. Consider the eigenvalue problem

$$\left((1 + x^4)y'\right)' + \lambda y = 0,\ x \in (0, 1),$$

$$y(0) = 0,\ y(1) + 2y'(1) = 0.$$

Then which of the following statements are true ?

(a) all the eigenvalues are negative.

(b) all the eigenvalues are positive.

(c) there exist some negative eigenvalues and some positive eigenvalues.

(d) there are no eigenvalues.

33. A possible initial strip $(x_0, y_0, z_0, p_0, q_0)$ for the Cauchy problem $pq = 1$, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and

$x_0(s) = s,\ y_0(s) = \frac{1}{s},\ z_0(s) = 1$ for $s > 1$ is :

(a) $\left(s, \frac{1}{s}, 1, \frac{1}{s}, s\right)$ (b) $\left(s, \frac{1}{s}, 1, -\frac{1}{s}, -s\right)$ (c) $\left(s, \frac{1}{s}, 1, \frac{1}{s}, -s\right)$ (d) $\left(s, \frac{1}{s}, 1, -\frac{1}{s}, s\right)$

34. Let $u(x, t)$ be the solution of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = xt, \qquad (-\infty < x < \infty,\ t > 0)$$

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0, \quad (-\infty < x < \infty)$$

Then $u(2, 3)$ is equal to:

(a) 9 (b) 1 (c) 27 (d) 12

35. The values of α, A, B, C for which the quadrature formula:

$$\int_{-1}^{1} (1-x)\, f(x)\, dx = Af(-\alpha) + Bf(0) + Cf(\alpha)$$

is exact for polynomials of highest possible degree, are:

(a) $\alpha = \sqrt{\frac{3}{5}},\ A = \frac{5}{9} + \frac{\sqrt{5}}{3\sqrt{3}},\ B = \frac{8}{9},\ C = \frac{5}{9} - \frac{\sqrt{5}}{3\sqrt{3}}$

(b) $\alpha = \sqrt{\frac{3}{5}},\ A = \frac{5}{9} - \frac{\sqrt{5}}{3\sqrt{3}},\ B = \frac{8}{9},\ C = \frac{5}{9} + \frac{\sqrt{5}}{3\sqrt{3}}$

(c) $\alpha = \sqrt{\frac{3}{5}},\ A = \frac{5}{9}\left(1 - \frac{\sqrt{3}}{\sqrt{5}}\right),\ B = \frac{8}{9},\ C = \frac{5}{9}\left(1 + \frac{\sqrt{3}}{\sqrt{5}}\right)$

(d) $\alpha = \sqrt{\frac{3}{5}},\ A = \frac{5}{9}\left(1 + \frac{\sqrt{3}}{\sqrt{5}}\right),\ B = \frac{8}{9},\ C = \frac{5}{9}\left(1 - \frac{\sqrt{3}}{\sqrt{5}}\right)$

36. Assume that h_1, h_2, g_1 and $g_2 \in C([a, b])$.

Let $\phi(x) = f(x) + \lambda \int_a^b \left[h_1(t)\, g_1(x) + h_2(t)\, g_2(x)\right] \phi(t)\, dt$

be an integral equation. Consider the following statements:

S_1: If the given integral equation has a solution for some $f \in C([a, b])$, then

$$\int_a^b f(t) g_1(t)\, dt = 0 = \int_a^b f(t) g_2(t)\, dt$$

S_2: The given integral equation has a unique solution for every $f \in C([a,b])$ if λ is not a characteristic number of the corresponding homogeneous equation.

Then,

(a) Both S_1 and S_2 are true (b) S_1 is true but S_2 is false
(c) S_1 is false but S_2 is true (d) Both S_1 and S_2 are false

37. The minimum value of the functional $I(y)=\int_0^{\pi}\left(\frac{dy}{dx}\right)^2 dx$, subject to $\int_0^{\pi} y^2(x)\,dx=1,\ y(0)=0=y(\pi)$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\frac{1}{3}$

38. Consider a mechanical system whose position is described using the generalized coordinates $q_1,\dots,q_n$. Let $T(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_n)$ be the kinetic energy of the system. If the generalized force $Q_j, 1\le j\le n$, acting on the system is zero, then the Lagrange equations of motion are:

(a) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=0, 1\le j\le n$ (b) $\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right)-\frac{\partial T}{\partial \dot{q}_j}=0, 1\le j\le n$

(c) $\frac{\partial}{\partial \dot{q}_j}\left(\frac{dT}{dt}\right)-2\frac{\partial T}{\partial q_j}=0, 1\le j\le n$ (d) $\frac{\partial}{\partial \dot{q}_j}\left(\frac{dT}{dt}\right)-\frac{\partial T}{\partial q_j}=0, 1\le j\le n$

39. Let y be a solution of $(1+x^2)y''+(1+4x^2)y=0,\ x>0;\ y(0)=0$. Then y has

(a) infinitely many zeros in [0, 1] (b) infinitely many zeros in $[1,\infty)$
(c) at least n zeros in $[0, n\pi]$, $\forall\, n\in\mathbb{N}$ (d) at most $3n$ zeros in $[0, n\pi]$, $\forall\, n\in\mathbb{N}$

40. Let $y=y(x)\in C^4([0,1])$ be an extremizing function for the functional $I(y)=\int_0^1\left[\left(\frac{d^2y}{dx^2}\right)^2-2y\right]dx$, satisfying $y(0)=0=y(1)$. Then an extremal $y(x)$, satisfying the given conditions at 0 and 1 together with the natural boundary conditions, is given by :

(a) $\frac{x}{24}(x-1)^3$ (b) $\frac{x^2}{24}(x-1)^2$ (c) $\frac{x}{24}(x^3-2x^2+1)$ (d) $\frac{x}{24}(x^3+x^2-2)$

41. The integral equation $\phi(x)=1+\frac{2}{\pi}\int_0^{\pi}(\cos^2 x)\phi(t)\,dt$ has

(a) no solution (b) unique solution
(c) more than one but finitely many solutions (d) infinitely many solutions

42. Consider the ordinary differential equation (ODE)

$$\begin{cases} y'(x)+y(x)=0, & x>0, \\ y(0)=1 \end{cases}$$

and the following numerical scheme to solve the ODE

$$\begin{cases} \dfrac{Y_{n+1}-Y_{n-1}}{2h}+Y_{n-1}=0, \quad n\geq 1, \\ Y_0=1, Y_1=1 \end{cases}$$

If $0<h<\frac{1}{2}$, then which of the following statements are true ?

(a) $(Y_n)\to\infty$ as $n\to\infty$

(b) $(Y_n)\to 0$ as $n\to\infty$

(c) (Y_n) is bounded

(d) $\max\limits_{nh\in[0,T]}|y(nh)-Y_n|\to\infty$ as $T\to\infty$

43. Consider the random effect model $y_{ij}=\mu+b_i+\varepsilon_{ij}$, $i=1,...,5;\ j=1,...,10$, where $b_i\sim$ i.i.d. $N(0,\tau^2)$ and $\varepsilon_{ij}\sim$ i.i.d. $N(0,\sigma^2)$ are all independent of each other. The parameter space for the model is $(\mu,\sigma^2,\tau^2)\in\mathbb{R}\times[0,\infty)\times[0,\infty)$. Let $\hat{\sigma}_u^2$ and $\hat{\tau}_u^2$ be the usual unbiased ANOVA estimators of σ^2 and τ^2 respectively, and $\hat{\sigma}_m^2$ and $\hat{\tau}_m^2$ be the maximum likelihood estimators of σ^2 and τ^2 respectively. Then, which of the following events can happen with positive probability for some parameter values ?

(a) $\hat{\sigma}_u^2$ is negative

(b) $\hat{\tau}_u^2$ is negative

(c) $\hat{\sigma}_m^2$ is negative

(d) $\hat{\tau}_m^2$ is negative

44. Let $\{X_n : n\geq 0\}$ be a Markov chain with state space $\mathbb{N}\cup\{0\}$ such that the transition probabilities are given by:

$$p_{ij}=\begin{cases} q & \text{for } j=0 \\ 1-q & \text{for } j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

for = 0, 1, 2,...., where $0<q<1$. Then which of the following statements are correct ?

(a) The Markov chain is irreducible.

(b) The Markov chain is aperiodic.

(c) $p_{00}^{(n)}=q$ for all $n\geq 1$

(d) The Markov chain is positive recurrent.

45. A random variable T has a symmetric distribution if T and $-T$ have the same distribution. Let X and Y be independent random variables. Then which of the following statements are correct ?

(a) If X and Y have the same distribution then $X-Y$ has a symmetric distribution.

(b) If $X\sim N(3,1)$ and $Y\sim N(2,2)$, then $2X-3Y$ has a symmetric distribution.

(c) If X and Y have the same symmetric distribution, then $X+Y$ has a symmetric distribution.

(d) If X has a symmetric distribution, then XY has a symmetric distribution.

46. Let $\{(X_n,Y_n):n\geq 1\}$ and (X,Y) be random variables, on (Ω,F,P). Then which of the following statements are correct ?

(a) If $X_n\to X$ almost surely, $Y_n\to Y$ almost surely, then $X_n+Y_n\to X+Y$ in distribution.

(b) If $X_n\to X$ in probability, $Y_n\to Y$ almost surely, then $X_n+Y_n\to X+Y$ in distribution.

(c) If $X_n\to X$ in probability, $Y_n\to Y$ in probability, then $X_n+Y_n\to X+Y$ in distribution.

(d) If $X_n\to X$ in distribution, $Y_n\to Y$ in distribution, then $X_n+Y_n\to X+Y$ in distribution.

47. Consider a Balanced Incomplete Block Design (v, b, r, k, λ) of v treatments and b blocks of k plots each. Let N be the $v \times b$ incidence matrix of the design. Then which of the following statements are correct ?

(a) $\lambda(k-1) = r(v-1)$ (b) $b \geq v$

(c) rank $(NN') = v$ (d) trace $(NN') = bk$

48. Suppose in a single service queue, customers arrive at a Poisson rate of one per ten minutes, and the service time is exponential at a rate of one service per five minutes. Let P_n be the probability that there are n customers in the system in steady state. Then which of the following statements are correct ?

(a) $P_{n+1} = \frac{1}{2} P_n$ for all $n \geq 0$

(b) The expected number of customers in the system is 1 in steady state.

(c) The expected number of customers in the system is 2 in steady state.

(d) The expected amount of time a customer spends in the system in steady state is 10 minutes.

49. Suppose the conditional p.d.f. of a random variable X given θ is,

$$f(x \mid \theta) = \begin{cases} 2x/\theta^2 & ; \quad 0 < x < \theta \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where the prior distribution of θ is uniform (0, 1). Based on a single observation x from X, which of the following statements are correct ?

(a) The Bayes estimates for θ under squared error loss function is $-x \log_e x$.

(b) The Bayes estimates for θ under squared error loss function is $-\frac{x \log_e x}{1-x}$.

(c) The Bayes estimates for θ under absolute error loss function is x.

(d) The Bayes estimates for θ under absolute error loss function is $\frac{2x}{1+x}$.

50. Let $\{X_i : i \geq 1\}$ be i.i.d. observations with $E(X_i) = 0$ and $\text{Var}(X_i) = \sigma^2 > 0$. Then which of the following statements are correct ?

(a) $\frac{1}{n}\sum_{i=1}^{n}(X_{2i-1} - X_{2i})^2$ is consistent for σ^2 (b) $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is consistent for σ^2

(c) $\frac{1}{n}\sum_{i=1}^{n}X_i^2$ is consistent for σ^2 (d) $\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is consistent for σ^2

51. Suppose that $M_{p \times p} \sim \text{Wishart}_p(I, m)$ and $Z \sim N_p(0, \Sigma)$ are independent where Σ is positive definite. Let $a \in \mathbb{R}^p$ be a fixed p-vector such that $a \neq 0$ and define $d = \frac{Z}{\|Z\|}$. Then which of the following random variables has a χ^2 distribution, possibly after being scaled by a constant factor ?

(a) $d'M^{-1}d$ (b) $(d'M^{-1}d)^{-1}$ (c) $a'Ma$ (d) $(a'Ma)^{-1}$

52. Let X_1, X_2 and X_3 be i.i.d. Normal random variables with mean θ and variance θ^2 where $\theta \in \mathbb{R}$ is unknown. Then which of the following statements are correct ?

(a) $\frac{X_1 + 2X_2 + 3X_3}{6}$ is unbiased for θ

(b) $\frac{X_1^2 + 4X_2^2 + 9X_3^2}{14}$ is unbiased for θ^2

(c) $\frac{2X_1 + X_3^2}{2}$ is unbiased for $\theta(1+\theta)$

(d) $X_2\left(1 - \frac{X_2}{2}\right)$ is unbiased for $\theta(1-\theta)$

53. Suppose $X_1, X_2, ..., X_n$ are i.i.d. Uniform $(\theta, 2\theta), \theta > 0$.

Let $X_{(1)} = \min\{X_1, ..., X_n\}$ and $X_{(n)} = \max\{X_1, ..., X_n\}$

Then which of the following statements are correct ?

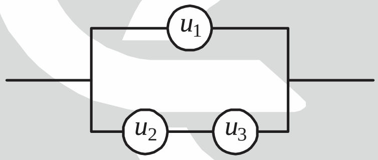
(a) $(X_{(1)}, X_{(n)})$ is jointly sufficient and complete for θ.

(b) $(X_{(1)}, X_{(n)})$ is jointly sufficient but not complete for θ.

(c) $\frac{X_{(n)}}{2}$ is a maximum likelihood estimator for θ.

(d) $X_{(1)}$ is a maximum likelihood estimator for θ.

54. Consider the following system with three independent components u_1, u_2 and u_3.



Suppose that the failure probability of each component is p, and let $f(p)$ be the probability that the whole system is still functioning. Then which of the following statements are correct ?

(a) $f\left(\frac{1}{2}\right) = \frac{5}{8}$ (b) $f\left(\frac{1}{2}\right) = \frac{3}{8}$ (c) $f\left(\frac{1}{3}\right) = \frac{22}{27}$ (d) $f\left(\frac{1}{4}\right) = \frac{50}{64}$

55. Given that $\{(x_i, y_i) : i = 1, 2, ..., n\}$ where $n \geq 2$ and not all x_i's are identical, the sample linear regression model $y = \alpha + \beta x + \varepsilon$ is fit. Let h_{ii} be the i-th diagonal element of the Hat matrix $H = X(X'X)^{-1}X'$ where $X_{n\times 2}$ is the corresponding model matrix. Then which of the following are possible for some choice of n and $x_1, x_2, ..., x_n$?

(a) $h_{ii} = -1$ for some i

(b) $h_{ii} = 0$ for some i

(c) $h_{ii} = 1$ for some i

(d) All h_{ii} are equal

56. Consider a Markov chain with state space S. Let $d(k)$ denote the period of state $k \in S$. Which of the following statements are correct ?

(a) For $i, j \in S$, if $\exists\, n, m > 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$ and i is recurrent, then j is recurrent.

(b) For $i, j \in S$, if $\exists\, n, m > 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$, then $d(i) = d(j)$.

(c) For $i,\ j \in S$, if $\exists\, r > 0$ such that $p_{ij}^{(r)} > 0$, then j cannot be transient.

(d) For $i,\ j \in S$, if $\exists\, r > 0$ such that $p_{ij}^{(r)} > 0$, and i is null recurrent then j is positive recurrent.

57. Let X be a discrete random variable with sample space $\chi = \{1, 2, ..., 10\}$ and probability mass function $p(x)$, $x \in \chi$. Consider testing the hypothesis.

$H_0 : p(x) = \frac{1}{10},\ x \in \chi$ against

$H_1 : p(x) \propto x,\ x \in \chi$

Based on a single observation X. Then which of the following statements are correct ?

(a) The test with critical region $\{X \geq 2\}$ is most powerful of its size.

(b) The test with critical region $\{X < 2\}$ is unbiased at level $a = 0.1$

(c) If $X = 7$ the p-value of the most powerful test is 0.6

(d) There exists a non-randomized test of size 0.05

58. Let X and Y be real-valued independent random variable on Ω. Then which of the following statements are correct ?

(a) $E[\cos(tX + uY)] = E[\cos(tX)]E[\cos(uY)] - E[\sin(tX)]E[\sin(uY)]$ for all $t, u \in \mathbb{R}$.

(b) If $X \sim N(2,1)$ and $Y \sim N(0,2)$, then $\mathrm{Var}\,(X + Y) = 3$ where $N(\mu, \sigma^2)$ represents normal distribution with mean μ and variance σ^2.

(c) $\{X = a\} \cap \{Y = b\} = \phi$ for all $a, b \in \mathbb{R}$

(d) $P(\{X = a\} \cap \{Y = b\}) = P(\{X = a\})\,P(\{Y = b\})$ for all $a, b \in \mathbb{R}$

59. Let $\{X_n : n \geq 1\}$ be i.i.d. with common unknown continuous distribution function $F(x - \theta)$, where θ is the unique median of F. Define, $Y_i = \begin{cases} 1 & \text{if} \quad X_i > 1 \\ 0 & \text{if} \quad X_i \leq 1 \end{cases}$ and $S_n = \sum_{i=1}^{n} Y_i$

For testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$, which of the following statements are correct ?

(a) $S_n \sim \text{Binomial}\left(n, \frac{1}{2}\right)$ under H_0

(b) Test based on S_n is distribution free under H_1

(c) Right-tailed test based on S_n is unbiased

(d) The sequence of right-tailed tests based on S_n, $n \geq 1$, is consistent

60. To estimate the population total $Y = \sum_{i=1}^{N} y_i$, where $y_1, y_2, ..., y_N$ are study variables and N is the size of the finite population, a sample of size n is drawn using $PPSWR(p_1, p_2, ..., p_N)$ scheme. If $\hat{Y}_{HH}$ is the Hansen-Hurwitz estimator for Y, then which of the following are correct ?

(a) $\hat{Y}_{HH}$ is unbiased for Y

(b) $\text{Var}(\hat{Y}_{HH}) = \frac{1}{n}\left\{\sum_{i=1}^{N} \frac{y_i^2}{p_i} - Y^2\right\}$

(c) $\text{Var}(\hat{Y}_{HH}) = \frac{1}{n}\sum_{i=1}^{N}\left\{\frac{y_i}{p_i} - E(\hat{Y}_{HH})\right\}^2 p_i$

(d) $\text{Var}(\hat{Y}_{HH}) = \frac{1}{n(n-1)}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^2 p_i p_j$

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(Answer Key) — PART - B (Mathematical Sciences)

1. (d)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)
6. (c)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (c)
11. (d)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (c)
16. (c)	**17.** (c)	**18.** (a)	**19.** (c)	**20.** (d)
21. (c)	**22.** (a)	**23.** (c)	**24.** (a)	**25.** (d)
26. (c)	**27.** (a)	**28.** (d)	**29.** (c)	**30.** (d)
31. (b)	**32.** (b)	**33.** (d)	**34.** (c)	**35.** (c)
36. (d)	**37.** (d)	**38.** (b)	**39.** (d)	**40.** (a)

(Answer Key) — PART - C (Mathematical Sciences)

1. (d)	**2.** (d)	**3.** (...)	**4.** (c)	**5.** (a)
6. (b), (c)	**7.** (c), (d)	**8.** (c)	**9.** (a), (c), (d)	**10.** (b), (d)
11. (c), (d)	**12.** (c)	**13.** (b), (d)	**14.** (a)	**15.** (a), (b)
16. (a), (b), (d)	**17.** (b), (c), (d)	**18.** (b), (c)	**19.** (a)	
20. (a), (b), (c), (d)				
21. (a), (c), (d)	**22.** (a), (b), (c)	**23.** (a), (b)	**24.** (a), (b)	**25.** (a), (d)
26. (a)	**27.** (b), (c), (d)	**28.** (c)	**29.** (a), (b), (c), (d)	**30.** (c), (d)
31. (a), (b), (c), (d)	**32.** (b)	**33.** (a), (b)	**34.** (a)	**35.** (a), (d)
36. (c)	**37.** (b)	**38.** (a), (c)	**39.** (b), (c)	**40.** (c)
41. (a)	**42.** (b), (c)			