CSIR-NET-(P YQ) MATHEMATICAL SCIENCE
JUNE-2023-I

PART-B

21. Suppose S is an infinite set. Assuming that the axiom of choice holds, which of the following is true?

(a) S is in bijection with the set of rational numbers.

(b) S is in bijection with the set of real numbers.

(c) S is in bijection with $S \times S$.

(d) S is in bijection with the power set of S.

Ans. (c)

22. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = (-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$. Which of the"following statements is true?

(a) The series is divergent.

(b) The series is convergent.

(c) The series is conditionally convergent.

(d) The series is absolutely convergent.

Ans. (b,c)

23. Let $x, y \in [0,1]$ be such that $x \neq y$ Which of the following statements is true for every $\in > 0$?

(a) There exists a positive integer N such that $|x-y| < 2^n \in$ for every integer $n \geq N$

(b) There exists a positive integer N such that $2^n \in < |x-y|$ for every integer $n \geq N$

(c) There exists a positive integer N such that $|x-y| < 2^{-n} \in$ for every integer $n \geq N$

(d) For every positive integer $N, |x-y| < 2^{-n} \in$ for some integer $n \geq N$

Ans. (a)

24. Which of the following assertions is correct?

(a) $\lim_n \sup e^{\cos\left(\frac{n\pi+(-1)^n 2e}{2n}\right)} > 1$

(b) $\lim_n e^{\log_e\left(\frac{n\pi^2+(-1)^n e^2}{7n}\right)}$ does not exist

(c) $\lim_n \inf e^{\sin\left(\frac{n\pi+(-1)^n 2e}{2n}\right)} < \pi$

(d) $\lim_n e^{\tan\left(\frac{n\pi^2+(-1)^n e^2}{7n}\right)}$ does not exist

Ans. (c)

25. How many real roots does the polynomial $x^3 + 3x - 2023$ have ?

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (b)

26. Which one of the following functions is uniformly continuous on the interval (0,1) ?

(a) $f(x)=\sin\frac{1}{x}$ (b) $f(x)=e^{-1/x^2}$

(c) $f(x)=e^x\cos\frac{1}{x}$ (d) $f(x)=\cos x\cos\frac{\pi}{x}$

Ans. **(b)**

27. Let $l\geq 1$ be a positive integer. What is the dimension of the $\mathbb{R}$-vector space of all polynomials in two variables over $\mathbb{R}$ having a total degree of at most l ?

(a) $l+1$ (b) $l(l-1)$ (c) $l(l+1)/2$ (d) $(l+1)(l+2)/2$

Ans. **(d)**

28. Let T be a linear operator on $\mathbb{R}^3$. Let $f(X)\in\mathbb{R}[X]$ denote its characteristic polynomial.

Consider the following statements.

(a) Suppose T is non-zero and 0 is an eigen value of T. If we write $f(X)=Xg(X)$ in $\mathbb{R}[X]$, then the linear operator $g(T)$ is zero.

(b) Suppose 0 is an eigenvalue of T with at least two linearly independent eigen vectors. If we write $f(X)=Xg(X)$ in $\mathbb{R}[X]$, then the linear operator $g(T)$ is zero. Which of the following is true ?

(a) Both (a) and (b) are true (b) Both (a) and (b) are false

(c) (a) is true and (b) is false (d) (a) is false and (b) is true

Ans. **(d)**

29. Let A be a 3×3 matrix with real entries. Which of the following assertions is FALSE?

(a) A must havea real eigenvalue

(b) If the determinant of A is 0, then 0 is an eigenvalue of A

(c) If the determinant of A is negative and 3 is an eigenvalue of A, then A must have three real eigenvalues.

(d) If the determinant of A is positive and 3 is an eigenvalue of A, then A must have three real eigenvalues

Ans. **(d)**

30. Let A be a 3×3 real matrix whose characteristic polynomial $p(T)$ is divisible by T^2.

Which of the following statements is true?

(a) The eigenspace of A for the eigenvalue 0 is two-dimensional.

(b) All the eigenvalues of A are real.

(c) $A^3=0$.

(d) A is diagonalizable

Ans. **(b)**

31. Let $x=(x_1,....,x_n)$ and $y=(y_1,....,y_n)$ denote vectors in $\mathbb{R}^n$ for a fixed $n\geq 2$. Which of the following defines an inner product on $\mathbb{R}^n$?

(a) $\langle x,y\rangle=\sum_{i,j=1}^{n}x_iy_j$ (b) $\langle x,y\rangle=\sum_{i,j=1}^{n}\left(x_i^2+y_j^2\right)$

(c) $\langle x,y\rangle=\sum_{j=1}^{n}j^3x_jy_j$ (d) $\langle x,y\rangle=\sum_{j=1}^{n}x_jy_{n-j+1}$

Ans. **(c)**

32. Consider the quadratic form Q(x, y, z) associated to the matrix

$$B=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & -2\end{bmatrix},\ \text{Let } S=\left\{\begin{bmatrix}a\\ b\\ c\end{bmatrix}\in\mathbb{R}^3 \mid Q(a,b,c)=0\right\}$$

Which of the following statements is FALSE?

(a) The intersection of S with the xy-plane is a line.

(b) The intersection of S with the xz-plane is an ellipse.

(c) S is the union of two planes.

(d) Q is a degenerate quadratic form

Ans. **(b)**

33. Let $f(z)=\exp\left(z+\frac{1}{z}\right), z\in\mathbb{C}\setminus\{0\}$. The residue of f at $z=0$ is

(a) $\sum_{l=0}^{\infty}\frac{1}{(l+1)!}$ (b) $\sum_{l=0}^{\infty}\frac{1}{l!(l+1)}$ (c) $\sum_{l=0}^{\infty}\frac{1}{l!(l+1)!}$ (d) $\sum_{l=0}^{\infty}\frac{1}{(l^2+1)!}$

Ans. **(c)**

34. Let f be an entire function that satisfies $|f(z)|\le e^y$ for all $z=x+iy\in\mathbb{C}$, where $x, y\in\mathbb{R}$.

Which of the following statements is true?

(a) $f(z)=ce^{-iz}$ for some $c\in\mathbb{C}$ with $|c|\le 1$

(b) $f(z)=ce^{iz}$ for some $c\in\mathbb{C}$ with $|c|\le 1$

(c) $f(z)=e^{-ciz}$ for some $c\in\mathbb{C}$ with $|c|\le 1$

(d) $f(z)=e^{ciz}$ for some $c\in\mathbb{C}$ with $|c|\le 1$

Ans. **(a)**

35. Consider the function f defined by $f(z)=\frac{1}{1-z-z^2}$ for $z\in\mathbb{C}$ such that $1-z-z^2\neq 0$

Which of the following statements is true ?

(a) f is an entire function.

(b) f has a simple pole at $z = 0$.

(c) f has a Taylor series expansion $f(z)=\sum_{n=0}^{\infty}a_n z^n$, where coefficients a_n, are recursively defined as follows: $a_0=1, a_1=0$ and $a_{n+2}=a_n+a_{n+1}$ for $n\ge 0$

(d) f has a Taylor series expansion $f(z)=\sum_{n=0}^{\infty}a_n z^n$, where coefficients a_n are recursively defined as follows: $a_0=1, a_1=1$ and $a_{n+2}=a_n+a_{n+1}$ for $n\ge 0$

Ans. **(d)**

36. Let C be the positively oriented circle in the complex plane of radius 3 centered at the origin.

What is the value of the integral $\int_C \frac{dz}{z^2\left(e^z - e^{-z}\right)}$?

(a) $\frac{i\pi}{12}$ (b) $\frac{-i\pi}{12}$ (c) $\frac{i\pi}{6}$ (d) $\frac{-i\pi}{6}$

Ans. **(d)**

37. Which of the following equations can occur as the class equation of a group of order10?

(a) 10 = 1 + 1 +....+ 1(10 times) (b) 10 =1 + 1 + 2 + 2 + 2 + 2

(c) 10 = 1 + 1 + 1 + 2 + 5 (d) 10 = 1 + 2 + 3 + 4

Ans. **(a)**

38. The number of solutions of the equation $x^2 = 1$ in the ring $\mathbb{Z}/105\mathbb{Z}$ is

(a) 0 (b) 2 (c) 4 (d) 8

Ans. **(d)**

39. Let p be a prime number. Let G be a group such that for each $g \in G$ there exists an $n \in \mathbb{N}$ such that $g^{p^n} = 1$. Which of the following statements is FALSE?

(a) If $|G| = p^6$, then G has a subgroup of index p^2

(b) If $|G| = p^6$, then G has at least five normal subgroups

(c) Center of G can be infinite

(d) There exists G with $|G| = p^6$ such that G has exactly six normal subgroups.

Ans. **(d)**

40. Consider $\mathbb{R}$ with the usual topology. Which of the following assertions is correct ?

(a) A finite set containing 33 elements has at least 3 different Hausdorff topologies.

(b) Let X be a non-empty finite set with a Hausdorff topology. Consider $X \times X$ with the product topology. Then, every function $f : X \times X \to \mathbb{R}$ is continuous.

(c) Let X bea discrete topological space having infinitely many elements. Let $f : \mathbb{R} \to X$ be a continuous function and $g : X \to \mathbb{R}$ be any non-constant“function. Then the range of $g \circ f$ contains at least 2 elements.

(d) If a non-empty metric space X has a finite dense subset, then there exists a discontinuous function $f : X > \mathbb{R}$.

Ans. **(b)**

41. Let $f : \mathbb{R}^2 \to \mathbb{R}$ bea locally Lipschitz function. Consider the initial value problem

$$\dot{x} = f(t, x),\ x(t_0) = x_0$$

for $(t_0, x_0) \in \mathbb{R}^2$. Suppose that $J(t_0, x_0)$ represents the maximal interval of existence for the initial value problem. Which of the following statements is true ?

(a) $J(t_0, x_0) = \mathbb{R}$ (b) $J(t_0, x_0)$ is an open set

(c) $J(t_0, x_0)$ is a closed set (d) $J(t_0, x_0)$ could be an empty set

Ans. **(b)**

42. Suppose $x(t)$ is the solution of the following initial value problem in $\mathbb{R}^2$?

$$\dot{x} = Ax,\ x(0) = x_0, \qquad \text{where } A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Which of the following statements is true ?

(a) $x(t)$ is a bounded solution for some $x_0 \neq 0$

(b) $e^{-6t}|x(t)| \to 0$ as $t \to \infty$, for all $x_0 \neq 0$

(c) $e^{-6t}|x(t)| \to \infty$ as $t \to \infty$, for all $x_0 \neq 0$

(d) $e^{-10t}|x(t)| \to 0$ as $t \to \infty$, for all $x_0 \neq 0$

Ans. (d)

43. Let $u(x, y)$ be the solution of the Cauchy problem.

$$uu_x + u_y = 0, \quad x \in \mathbb{R},\ y > 0,$$
$$u(x,0) = x, \quad x \in \mathbb{R}.$$

Which of the following is the value of $u(2,3)$?

(a) 2 (b) 3 (c) 1/2 (d) 1/3

Ans. (c)

44. Let $u(x, t)$ be the solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0 & 0 < x < 2, t > 0, \\ u(0,t) = 0 &= u(2,t), & \forall t > 0, \\ u(x,0) &= \sin(\pi x) + 2\sin(2\pi x), & 0 \le x \le 2, \\ u_t(x,0) &= 0, & 0 \le x \le 2. \end{aligned}$$

Which of the following is true?

(a) $u(1,1) = -1$ (b) $u(1/2,1) = 0$ (c) $u(1/2,2) = 1$ (d) $u(1/2,1/2) = \pi$

Ans. (c)

45. Which of the following values of a, b, c and d will produce a quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3 ?

(a) $a = 1, b = 1, c = \frac{1}{3}, d = -\frac{1}{3}$

(b) $a = -1, b = 1, c = \frac{1}{3}, d = -\frac{1}{3}$

(c) $a = 1, b = 1, c = -\frac{1}{3}, d = \frac{1}{3}$

(d) $a = 1, b = -1, c = \frac{1}{3}, d = -\frac{1}{3}$

Ans. (a)

46. Consider the variational problem (P)

$$J(y(x)) = \int_0^1 \left[(y')^2 - y|y|y' + xy \right] dx, \quad y(0) = 0,\ y(1) = 0$$

Which of the following statements is correct ?

(a) (P) has no stationary function (extremal)

(b) $y \equiv 0$ is the only stationary function (extremal) for (P)

(c) (P) has a unique stationary function (extremal) y not identically equal to 0

(d) (P) has infinitely many stationary functions (extremal)

Ans. **(c)**

47. For the unknown $y: [0,1] \to \mathbb{R}$, consider the following two-point boundary value problem:

$$\begin{cases} y''(x) + 2y(x) & = 0 \text{ for } x \in (0,1) \\ y(0) & = y(1) = 0 \end{cases}$$

It is given that the above boundary value problem corresponds to the following integral equation:

$$y(x) = 2\int_0^1 K(x,t)\, y(t)\, dt \quad \text{for } x \in [0,1]$$

Which of the following is the kernel $K(x, t)$?

(a) $K(x,t) = \begin{cases} t(1-x) & \text{for } t < x \\ x(1-t) & \text{for } t > x \end{cases}$

(b) $K(x,t) = \begin{cases} t^2(1-x) & \text{for } t < x \\ x^2(1-t) & \text{for } t > x \end{cases}$

(c) $K(x,t) = \begin{cases} \sqrt{t}(1-x) & \text{for } t < x \\ \sqrt{x}(1-t) & \text{for } t > x \end{cases}$

(d) $K(x,t) = \begin{cases} \sqrt{t^3}(1-x) & \text{for } t < x \\ \sqrt{x^3}(1-t) & \text{for } t > x \end{cases}$

Ans. **(a)**

48. Consider the constants a and b such that the following generalized coordinate transformation from (p, q) to (P, Q) is canonical.

$$Q = pq^{(a+1)},\ P = q^b$$

What are the values of a and b ?

(a) $a = -1, b = 0$ (b) $a = -1, b = 1$ (c) $a = 1, b = 0$ (d) $a = 1, b = -1$

Ans. **(d)**

49. If $f(x)$ is a probability density on the real line, then which of the following is NOT a valid probability density?

(a) $f(x + 1)$ (b) $f(2x)$ (c) $2f(2x - 1)$ (d) $3x^2 f(x^3)$

Ans. **(b)**

50. Which of the following is a valid cumulative distribution function ?

(a) $F(x) = \begin{cases} \dfrac{1}{2+x^2} & \text{if } x < 0 \\ \dfrac{2+x^2}{3+x^2} & \text{if } x \geq 0 \end{cases}$

(b) $F(x) = \begin{cases} \dfrac{1}{2+x^2} & \text{if } x < 0 \\ \dfrac{2+x^2}{3+2x^2} & \text{if } x \geq 0 \end{cases}$

(c) $F(x) = \begin{cases} \dfrac{1}{2+x^2} & \text{if } x < 0 \\ \dfrac{2\cos(x)+x^2}{4+x^2} & \text{if } x \geq 0 \end{cases}$

(d) $F(x) = \begin{cases} \dfrac{1}{2+x^2} & \text{if } x < 0 \\ \dfrac{1+x^2}{4+x^2} & \text{if } x \geq 0 \end{cases}$

Ans. **(a)**

PART-C

61. Let $\{x_n\}$ be a sequence of positive real numbers. If $\sigma_n = \frac{1}{n}(x_1 + x_2 + \ldots x_n)$, then which of the following are true? (Here lim sup denotes the limit supremum of a sequence.)

(a) If $\limsup\{x_n\} = \ell$ and $\{x_n\}$ is decreasing, then $\limsup\{\sigma_n\} = \ell$

(b) $\limsup\{x_n\} = \ell$ and only if $\limsup\{\sigma_n\} = \ell$

(c) If $\limsup\left\{n\left(\frac{x_n}{x_{(n+1)}} - 1\right)\right\} < 1$, then $\Sigma_n x_n$ is convergent.

(d) If $\limsup\left\{n\left(\frac{x_n}{x_{(n+1)}} - 1\right)\right\} < 1$, then $\Sigma_n x_n$ is divergent

Ans. **(a,d)**

62. Under which of the following conditions is the sequence $\{x_n\}$ of real numbers convergent ?

(a) The subsequences $\{x_{(2n+1)}\}, \{x_{2n}\}$ and $\{x_{3n}\}$ are convergent and have the same limit.

(b) The subsequences $\{x_{(2n+1)}\}, \{x_{2n}\}$ and $\{x_{3n}\}$ are convergent.

(c) The subsequences $\{x_{kn}\}_n$, are convergent for every $k \geq 2$.

(d) $\lim_n \left|x_{(n+1)} - x_n\right| = 0$

Ans. **(a,b)**

63. Which of the following are true?

(a) For $n \geq 1$, the sequence of functions $f_n : (0,1) \to (0,1)$ defined by $f_n(x) = x^n$ is uniformly convergent.

(b) For $n \geq 1$, the sequence of functions $f_n : (0,1) \to (0,1)$ defined by $f_n(x) = \frac{x^n}{\log(n+1)}$ is uniformly convergent

(c) Fo r $n \geq 1$, the sequence of functions $f_n : (0,1) \to (0,1)$ defined by $f_n(x) = \frac{x^n}{1+x^n}$ uniformly convergent.

(d) For $n \geq 1$, the sequence of functions $f_n : (0,1) \to (0,1)$ defined by $f_n(x) = \frac{x^n}{1+nx^n}$ is not uniformly convergent

Ans. **(b)**

64. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin(\pi / x) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

On which of the following subsets of $\mathbb{R}$, the restriction of f is a continuous function ?

(a) $[-1, 1]$

(b) $(0,1)$

(c) $\{0\} \cup \{(1/n) : n \in \mathbb{N}\}$

(d) $\{1/2^n : n \in \mathbb{N}\}$

Ans. **(b,c,d)**

65. Define $f:\mathbb{R}^4 \to \mathbb{R}$ by $f(x,y,z,w) = xw - yz$. Which of the following statements are true?

(a) f is continuous

(b) if $U = \{(x,y,z,w) \in \mathbb{R}^4 : xy + zw = 0,\ x^2 + z^2 = 1,\ y^2 + w^2 = 1\}$ then f is uniformly continuous on U.

(c) if $V = \{(x,y,z,w) \in \mathbb{R}^4 : x = y, z = w\}$, then f is uniformly continuous on V.

(d) if $W = \{(x,y,z,w) \in \mathbb{R}^4 : 0 \le x + y + z + w \le 1\}$, then f is unbounded on W.

Ans. **(a,b,c,d)**

66. Consider the following statements:

(a) Let f be a continuous function on $[1,\infty)$ taking non-negative values such that $\int_1^\infty f(x)\,dx$ converges.

Then $\Sigma_{n\ge1} f(n)$ converges.

(b) Let f be a function on $[1,\infty)$ taking non-negative values such that $\int_1^\infty f(x)\,dx$ converges.

Then $\lim_{x\to\infty} f(x) = 0$

(c) Let f be a continuous, decreasing function on $[1,\infty)$ taking non-negative values such that $\int_1^\infty f(x)\,dx$ does not converge. Then $\Sigma_{n\ge1} f(n)$ does not converge.

Which of the following options are true?

(a) (a), (b) and (c) all are true
(b) Both (a) and (b) are false.
(c) (c) is true
(d) (b) is true

Ans. **(b,c)**

67. Let μ denote the Lebesgue measure on $\mathbb{R}$ and $\mu*$ be the associated Lebesgue outer measure. Let A be a non-empty subset of [0,1]. Which of the following statements are true ?

(a) If both interior and closure of A have the same outer measure, then A is Lebesgue measurable.

(b) If A is open, then A is Lebesgue measurable and $\mu(A) > 0$.

(c) If A is not Lebesgue measurable, then the set of irrationals in A must have positive outer measure.

(d) If $\mu^*(A) = 0$, then A has empty interior.

Ans. **(a,b,c,d)**

68. Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = x^2 - y^3$$

Which of the following statements are true ?

(a) There is no continuous real-valued function g defined on any interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.

(b) There is exactly one continuous real-valued function g defined on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.

(c) There is exactly one differentiable real-valued function g defined on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.

(d) There are two distinct differentiable real-valued functions g on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.

Ans. **(b)**

69. Suppose that $f:\mathbb{R}^n \to \mathbb{R}, n \geq 2$ is a C^2 function satisfying

$$f(y) \geq f(x) + \nabla(f)(x)(y-x)$$

for every x, y in $\mathbb{R}^n$. Here Δ denotes the gradient. Which of the following statements are true ?

(a) f is constant

(b) f is convex

(c) f is convex and bounded

(d) f is constant if f is bounded

Ans. (b,d)

70. Which of the following statements are true for an arbitrary normed linear space U ?

(a) Every bounded linear functional from U to R is continuous

(b) U is isomorphic to its double-dual U^{**}

(c) Forevery $x \in U$, we have $\|x\| = \sup_{\|f\| \leq 1} |f(x)|$, where f denotes elements of U

(d) The closed unit ball in U is compact

Ans. (a,c)

71. Let V be the vector space of all polynomials in one variable of degree at most 10 with real coefficients.

Let W_1 be the subspace of V consisting of polynomials of degree at most 5 and let W_2 be the subspace of V consisting of polynomials such that the sum of their coefficients is 0. Let W be the smallest subspace of V containing both W_1 and W_2.

Which of the following statements are true?

(a) The dimension of W is at most 10

(b) $W = V$

(c) $W_1 \subset W_2$

(d) The dimension of $W_1 \cap W_2$ is at most 5

Ans. (b,d)

72. Let B bea 3×5 matrix with entries from $\mathbb{Q}$. Assume that $\{v \in \mathbb{R}^5 \mid Bv = 0\}$ is a three dimensional real vector space. Which of the following statements are true?

(a) $\{v \in \mathbb{Q}^5 \mid Bv = 0\}$ is a three-dimensional vector space over $\mathbb{Q}$.

(b) The linear transformation T: $\mathbb{Q}^3 > \mathbb{Q}^5$ given by $T(v) = B^t v$ is injective.

(c) The column span of B is two-dimensional.

(d) The linear transformation T: $\mathbb{Q}^3 \to \mathbb{Q}^3$ given by $T(v) = BB v^t$ is injective

Ans. (a,c)

73. Let V be a finite dimensional real vector space and T_1, T_2 be two nilpotent operators on V.

Let $W_1 = \{v \in V : T_1(v) = 0\}$ and $W_2 = \{v \in V : T_2(v) = 0\}$. Which of the“following statements are FALSE?

(a) If T_1 and T_2 are similar, then W_1 and W_2 are isomorphic vector spaces.

(b) If W_1 and W_2 are isomorphic vector spaces, then T_1 and T_2 have the same minimal polynomial

(c) If $W_1 = W_2 = V$, then T_1 and T_2 are similar.

(d) If W_1 and W_2 are isomorphic, then T_1 and T_2 have the same characteristic polynomial

Ans. (b)

74. Let V be the real vector space of real polynomials in one variable with degree less than or equal to 10 (including the zero polynomial). Let T: $V \to V$ be the linear map defined by $T(p) = p'$, where p' denotes the derivative of p. Which of the following statements“are correct?

(a) rank (T) = 10

(b) determinant (T) = 0

(c) trace(T) = 0

(d) All the eigenvalues of T are equal to 0

Ans. (a,b,c,d)

75. Suppose A is a 5 × 5 block diagonal real matrix with diagonal blocks

$$\begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix}, \begin{pmatrix} e & 1 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}$$

Which of the following statements are true ?

(a) The algebraic multiplicity of e in A is 5

(b) A is not diagonalisable

(c) The geometric multiplicity of e in A is 3

(d) The geometric multiplicity of e in A is 2

Ans. **(a,b,c)**

76. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ bea linear transformation satisfying $T^3 - 3T^2 = -2I$, where I: $\mathbb{R}^3 \to \mathbb{R}^3$ is the identity transformation . Which of the following statements are true?

(a) $\mathbb{R}^3$ must admit a basis B_1 such that the matrix of T with respect to B_1 is symmetric

(b) $\mathbb{R}^3$ must admit a basis B_2 such that the matrix of T with respect to B_2 is upper triangular

(c) $\mathbb{R}^3$ must contain a non-zero vector v such that $Tv = v$

(d) $\mathbb{R}^3$ must contain two linearly independent vectors v_1, v_2 such that $Tv_1 = v_1$ and $Tv_2 = v_2$

Ans. **(a,b)**

77. Let V be an inner product space and let $v_1, v_2, v_3 \in V$ be an orthogonal set of vectors. Which of the following statements are true?

(a) The vectors $v_1 + v_2 + 2v_3, v_2 + v_3, v_2 + 3v_3$ can be extended to a basis of V

(b) The vectors $v_1 + v_2 + 2v_3, v_2 + v_3, v_2 + 3v_3$ can be extended to an orthogonal basis of V

(c) The vectors $v_1 + v_2 + 2v_3, v_2 + v_3, 2v_1 + v_2 + 3v_3$ can be extended to a basis of V

(d) The vectors $v_1 + v_2 + 2v_3, 2v_1 + v_2 + v_3, 2v_1 + v_2 + 3v_3$ can be extended to a basis of V

Ans. **(Droppd)**

78. Consider the following quadratic forms over $\mathbb{R}$

(a) $6X^2 - 13XY + 6Y^2$

(b) $X^2 - XY + 2Y^2$

(c) $X^2 - XY - 2Y^2$.

Which of the following statements are true?

(a) Quadratic forms (a) and (b) are equivalent

(b) Quadratic forms (a) and (c) are equivalent

(c) Quadratic form (5) is positive definite

(d) Quadratic form (c) is positive definite

Ans. **(b,c)**

79. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and C the positively oriented boundary $\{|z| = 1\}$. Fix a finite set $\{z_1, z_2, ..., z_n\} \subseteq \mathbb{D}$ of distinct points and consider the polynomial $g(z) = (z - z_1)(z - z_2)....(z - z_n)$ of degree n. Let f be a holomorphic function in an open neighbourhood of $\overline{\mathbb{D}}$ and define

$$P(z) = \frac{1}{2\pi i}\int_C f(\zeta)\frac{g(\zeta) - g(z)}{(\zeta - z)g(\zeta)}d\zeta$$

Which of the following statements are true ?

(a) P is a polynomial of degree n

(b) P is a polynomial of degree $n-1$

(c) P is a rational function on C with poles at $z_1, z_2, ..., z_n$

(d) $P(z_j) = f(z_j)$ for $j = 1, 2, ..., n$

Ans. (d)

80. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Consider the following statements

(a) $f : D \to D$ be a holomorphic function. Suppose α, β are distinct complex numbers in D such that $f(\alpha) = \alpha$ and $f(\beta) = \beta$. Then $f(z) = z$ for all $z \in D$.

(b) There does not exist a bijective holomorphic function from D to the set of all complex numbers whose imaginary part is positive.

(c) $f : D \to D$ be a holomorphic function. Suppose $\alpha \in D$ be such that $f(\alpha) = \alpha$ and $f'(\alpha) = 1$. Then $f(z) = z$ for all $z \in D$

Which of the following options are true ?

(a) (a), (b) and (c) are all true

(b) (a) is true.

(c) Both (a) and (b) are false

(d) Both (a) and (c) are true.

Ans. (b,d)

81. Let $f : \{z : |z| < 1\} \to \{z : |z| \le 1/2\}$ bea holomorphic function such that $f(0) = 0$. Which of the following statements are true?

(a) $|f(z)| \le |z|$ for all z in $\{z : |z| < 1\}$

(b) $|f(z)| \le \left|\frac{z}{2}\right|$ for all z in $\{z : |z| < 1\}$

(c) $|f(z)| \le 1/2$ for all z in $\{z : |z| < 1\}$

(d) It is possible that $f(1/2) = 1/2$.

Ans. (a,b,c)

82. Let $f(z)$ be an entire function on C. Which of the following statements are true?

(a) $f(\bar{z})$ is an entire function

(b) $\overline{f(z)}$ is an entire function

(c) $\overline{f(\bar{z})}$ is an entire function

(d) $\overline{f(z)} + f(\bar{z})$ is an entire function

Ans. (c)

83. Which of the following statements are correct ?

(a) If G is a group of order 244, then G contains a unique subgroup of order 27

(b) If G is a group of order 1694, then G contains a unique subgroup of order 121

(c) There exists a group of order 154 which contains a unique subgroup of order 7

(d) There exists a group of order 121 which contains two subgroups of order 11

Ans. (b,c,d)

84. Let G be a group of order 2023. Which of the following statements are true?

(a) G is an Abelian group

(b) G is a cyclic group

(c) G is a simple group

(d) G is not a simple group

Ans. (a,d)

85. Let G_1 and G_2 be two groups and $\varphi: G_1 \to G_2$ be a surjective group homomorphism. Which of the following statements are true?

(a) If G_1 is cyclic then G_2 is cyclic

(b) If G_1 is Abelian then G_2 is Abelian

(c) If H is a subgroup of G_1 then $\varphi(H)$ is a subgroup of G_2

(d) If N is anormal subgroup of G_1 then $\varphi(N)$ is a normal subgroup of G_2

Ans. **(a,b,c,d)**

86. Let $n \geq 1$ be a positive integer and S_n, the symmetric group on n symbols. Let $\Delta = \{(g, g): g \in S_n\}$. Which of the following statements are necessarily true?

(a) The map $f: S_n \times S_n \to S_n$ given by $f(a, b) = ab$ isa group homomorphism.

(b) Δ is a subgroup of $S_n \times S_n$.

(d) Δ is a normal subgroup of $S_n \times S_n$.

(d) Δ is anormal subgroup of $S_n \times S_n$, if n is a prime number

Ans. **(b)**

87. Which of the following are maximal ideals of $\mathbb{Z}[X]$?

(a) Ideal generated by 2 and $(1 + X^2)$

(b) Ideal generated by 2 and $(1 + X + X^2)$

(c) Ideal generated by 3 and $(1 + X^2)$

(d) Ideal generated by 3 and $(1 + X + X^2)$

Ans. **(b,c)**

88. Let E be a finite algebraic Galois extension of F with Galois group G. Which of the following statements are true ?

(a) There is an intermediate field K with $K \neq F$ and $K \neq E$ such that K is a Galois extension of F.

(b) If every proper intermediate field K is a Galois extension of F then G is Abelian.

(c) If E has exactly three intermediate fields including F and E then G is Abelian

(d) If $[E: F] = 99$ then every intermediate field is a Galois extension of F

Ans. **(c,d)**

89. Which of the following statements are correct ?

(a) The set of open right half-planes is a basis for the usual (Euclidean) topology on $\mathbb{R}^2$?

(b) The set of lines parallel to Y-axis is a basis for the dictionary order topology on $\mathbb{R}^2$.

(c) The set of open rectangles is a basis for the usual (Euclidean) topology on $\mathbb{R}^2$?

(d) The set of line segments (without end points) parallel to Y-axis is a basis for the dictionary order topology on $\mathbb{R}^2$?

Ans. **(c,d)**

90. Let $X = \prod_{n=1}^{\infty}[0,1]$, that is, the space of sequences $\{x_n\}_{n\geq 1}$ with $x_n \in [0,1], n \geq 1$.

Define the metric $d: X \times X \to [0, \infty]$ by

$$d\left(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}\right) = \sup_{n\geq 1} \frac{|x_n - y_n|}{2^n}$$

Which of the following statements are true?

(a) The metric topology on X is finer than the product topology on X

(b) The metric topology on X is coarser than the product topology on X

(c) The metric topology on X is same as the product topology on X

(d) The metric topology on X and the product topology on X are not comparable.

Ans. **(a,b,c)**

91. Let $f \in C^1(\mathbb{R})$ be bounded. Let us consider the initial-value problem

$$(P)\begin{cases} x'(t) = f(x(t)), & t > 0 \\ x(0) = 0 \end{cases}$$

Which of the following statements are true?

(a) (P) has solution(s) defined for all $t > 0$

(b) (P) has a unique solution.

(c) (P) has infinitely many solutions

(d) The solution(s) of (P) is/are Lipschitz

Ans. **(a,b,d)**

92. Consider the following initial value problem (IVP)

$$\frac{du}{dt} = t^2 y^{1/5}, u(0) = 0$$

Which of the following statements are correct ?

(a) The function $g(t,u) = t^2 u^{1/5}$ does not satisfy the Lipschitz's condition with respect to u in any neighbourhood of $u = 0$

(b) There is no solution for the IVP.

(c) There exist more than one solution for the IVP.

(d) The function $g(t,u) = t^2 u^{1/5}$ satisfies the Lipschitz's condition with respect to u in some neighbourhood of $u = 0$ and hence there exists a unique solution for the IVP

Ans. **(a,c)**

93. Let us consider the following two initial value problems

$$(P) = \begin{cases} x'(t) = \sqrt{x(t)}, & t > 0 \\ x(0) = 0, \end{cases}$$

and

$$(Q) = \begin{cases} y'(t) = -\sqrt{y(t)}, & t > 0 \\ y(0) = 0. \end{cases}$$

Which of the following statements are true ?

(a) (P) has a unique solution in $[0, \infty)$

(b) (Q) has a unique solution in $[0, \infty)$

(c) (P) has infinitely many solutions in $[0, \infty)$

(d) (Q) has infinitely many solutions in $[0, \infty)$

Ans. **(b,c)**

94. Let $u: \mathbb{R}^2 \to \mathbb{R}$ be the solution to the Cauchy problem:

$$\begin{cases} \partial_x u + 2\partial_y u = 0 & \text{for } (x,y) \in \mathbb{R}^2 \\ u(x,y) = \sin(x) & \text{for } y = 3x+1, x \in \mathbb{R} \end{cases}$$

Let $v: \mathbb{R}^2 \to \mathbb{R}$ be the solution to the Cauchy problem:

$$\begin{cases} \partial_x v + 2\partial_y v = 0 & \text{for } (x,y) \in \mathbb{R}^2 \\ v(x,0) = \sin(x) & \text{for } x \in \mathbb{R} \end{cases}$$

Let $\delta = [0,1]\times[0,1]$

Which of the following statements are true ?

(a) u changes sign in the interior of δ

(b) $u(x, y) = v(x, y)$ along a line in δ

(c) v changes sign in the interior of δ

(d) v vanishes along a line in δ

Ans. (b,c,d)

95. Let $u = u(x, y)$ be the solution to the following Cauchy problem

$$u_x + u_y = e^u \text{ for } (x,y)\in \mathbb{R}\times\left(0,\frac{1}{e}\right) \text{ and } u(x,0)=1 \text{ for } x\in\mathbb{R}$$

Which of the following statements are true ?

(a) $u\left(\frac{1}{2e},\frac{1}{2e}\right)=1$

(b) $u_x\left(\frac{1}{2e},\frac{1}{2e}\right)=0$

(c) $u_y\left(\frac{1}{4e},\frac{1}{4e}\right)=\log 4$

(d) $u_y\left(0,\frac{1}{4e}\right)=\frac{4e}{3}$

Ans. (b,d)

96. Consider the following two sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_n = \frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}, \quad b_n = \frac{1}{n}$$

Which of the following statements are true?

(a) $\{a_n\}$ converges to log 2, and has the same convergence rate as the sequence $\{b_n\}$

(b) $\{a_n\}$ converges to log 4, and has the same convergence rate as the sequence $\{b_n\}$

(c) $\{a_n\}$ converges to log 2, but does not have the same convergence rate as the“sequence $\{b_n\}$.

(d) $\{a_n\}$ does not converge.

Ans. (a)

97. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=\frac{1}{4}+x-x^2$. Given $a\in\mathbb{R}$, let us define the sequence $\{x_n\}$ by $x_0 = a$ and $x_n = f(x_{n-1})$ for $n\geq 1$

Which of the following statements are true?

(a) If $a = 0$, then the sequence $\{x_n\}$ converges to $\frac{1}{2}$ –

(b) If $a = 0$, then the sequence $\{x_n\}$ converges to $-\frac{1}{2}$

(c) The sequence $\{x_n\}$ converges for every $a\in\left(-\frac{1}{2},\frac{3}{2}\right)$ and it converges to $\frac{1}{2}$

(d) If $a = 0$, then the sequence $\{x_n\}$ does not converge.

Ans. (a,c)

98. Suppose $y(x)$ is an extremal of the following functional

$$J\left(y(x)\right)=\int_0^1\left(y(x)^2-4y(x)y'(x)+4y'(x)^2\right)dx$$

subject to $y(0) = 1$ and $y'(0) = 1/2$.

Which of the following statements are true?

(a) y is a convex function.

(b) y is a concave function

(c) $y(x_1+x_2)=y(x_1)y(x_2)$ for all x_1, x_2 in [0, 1]

(d) $y(x_1x_2)=y(x_1)+y(x_2)$ for all x_1, x_2 $[0,1]$

Ans. (a,c)

99. Let $y(x)$ and $z(x)$ be the stationary functions (extremals) of the variational problem

$$J\left(y(x),z(x)\right)=\int_0^1\left[(y')^2+(z')^2+y'z'\right]dx$$

subject to $y(0) = 1$, $y(1) = 0$, $z(0) = -1$, $z(1) = 2$

Which of the following statements are correct?

(a) $z(x) + 3y(x) = 2$ for $x\in[0,1]$

(b) $3z(x) + y(x) = 2$ for $x\in[0,1]$

(c) $y(x) + z(x) = 2x$ for $x\in[0,1]$

(d) $y(x) + z(x) = x$ for $x\in[0,1]$

Ans. (a,c)

100. Let $\lambda_1 < \lambda_1$ be two real characteristic numbers for the following homogeneous"integral equation:

$$\varphi(x)=\lambda\int_0^{2\pi}\sin(x+t)\varphi(t)\,dt$$

and let $\mu_1 < \mu_2$ be two real characteristic numbers for the following homogeneous"integral equation:

$$\psi(x)=\mu\int_0^{\pi}\cos(x+t)\psi(t)\,dt$$

Which of the following statements are true?

(a) $\mu_1<\lambda_1<\lambda_2<\mu_2$

(b) $\lambda_1<\mu_1<\mu_2<\lambda_2$

(c) $|\mu_1-\lambda_1|=|\mu_2-\lambda_2|$

(d) $|\mu_1-\lambda_1|=2|\mu_2-\lambda_2|$

Ans. (a,c)