

**Solutions to the 68th William Lowell Putnam Mathematical Competition**  
**Saturday, December 1, 2007**

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A-1 The only such  $\alpha$  are  $2/3, 3/2, (13 \pm \sqrt{601})/12$ .

**First solution:** Let  $C_1$  and  $C_2$  be the curves  $y = \alpha x^2 + \alpha x + \frac{1}{24}$  and  $x = \alpha y^2 + \alpha y + \frac{1}{24}$ , respectively, and let  $L$  be the line  $y = x$ . We consider three cases.

If  $C_1$  is tangent to  $L$ , then the point of tangency  $(x, x)$  satisfies

$$2\alpha x + \alpha = 1, \quad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry,  $C_2$  is tangent to  $L$  there, so  $C_1$  and  $C_2$  are tangent. Writing  $\alpha = 1/(2x + 1)$  in the first equation and substituting into the second, we must have

$$x = \frac{x^2 + x}{2x + 1} + \frac{1}{24},$$

which simplifies to  $0 = 24x^2 - 2x - 1 = (6x + 1)(4x - 1)$ , or  $x \in \{1/4, -1/6\}$ . This yields  $\alpha = 1/(2x + 1) \in \{2/3, 3/2\}$ .

If  $C_1$  does not intersect  $L$ , then  $C_1$  and  $C_2$  are separated by  $L$  and so cannot be tangent.

If  $C_1$  intersects  $L$  in two distinct points  $P_1, P_2$ , then it is not tangent to  $L$  at either point. Suppose at one of these points, say  $P_1$ , the tangent to  $C_1$  is perpendicular to  $L$ ; then by symmetry, the same will be true of  $C_2$ , so  $C_1$  and  $C_2$  will be tangent at  $P_1$ . In this case, the point  $P_1 = (x, x)$  satisfies

$$2\alpha x + \alpha = -1, \quad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing  $\alpha = -1/(2x + 1)$  in the first equation and substituting into the second, we have

$$x = -\frac{x^2 + x}{2x + 1} + \frac{1}{24},$$

or  $x = (-23 \pm \sqrt{601})/72$ . This yields  $\alpha = -1/(2x + 1) = (13 \pm \sqrt{601})/12$ .

If instead the tangents to  $C_1$  at  $P_1, P_2$  are not perpendicular to  $L$ , then we claim there cannot be any point where  $C_1$  and  $C_2$  are tangent. Indeed, if we count intersections of  $C_1$  and  $C_2$  (by using  $C_1$  to substitute for  $y$  in  $C_2$ , then solving for  $y$ ), we get at most four solutions counting multiplicity. Two of these are  $P_1$  and  $P_2$ , and any point of tangency counts for two more. However, off of  $L$ , any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible  $\alpha$ .

**Second solution:** For any nonzero value of  $\alpha$ , the two conics will intersect in four points in the complex projective plane  $\mathbb{P}^2(\mathbb{C})$ . To determine the  $y$ -coordinates of these intersection points, subtract the two equations to obtain

$$(y - x) = \alpha(x - y)(x + y) + \alpha(x - y).$$

Therefore, at a point of intersection we have either  $x = y$ , or  $x = -1/\alpha - (y + 1)$ . Substituting these two possible linear conditions into the second equation shows that the  $y$ -coordinate of a point of intersection is a root of either  $Q_1(y) = \alpha y^2 + (\alpha - 1)y + 1/24$  or  $Q_2(y) = \alpha y^2 + (\alpha + 1)y + 25/24 + 1/\alpha$ .

If two curves are tangent, then the  $y$ -coordinates of at least two of the intersection points will coincide; the converse is also true because one of the curves is the graph of a function in  $x$ . The coincidence occurs precisely when either the discriminant of at least one of  $Q_1$  or  $Q_2$  is zero, or there is a common root of  $Q_1$  and  $Q_2$ . Computing the discriminants of  $Q_1$  and  $Q_2$  yields (up to constant factors)  $f_1(\alpha) = 6\alpha^2 - 13\alpha + 6$  and  $f_2(\alpha) = 6\alpha^2 - 13\alpha - 18$ , respectively. If on the other hand  $Q_1$  and  $Q_2$  have a common root, it must be also a root of  $Q_2(y) - Q_1(y) = 2y + 1 + 1/\alpha$ , yielding  $y = -(1 + \alpha)/(2\alpha)$  and  $0 = Q_1(y) = -f_2(\alpha)/(24\alpha)$ .

Thus the values of  $\alpha$  for which the two curves are tangent must be contained in the set of zeros of  $f_1$  and  $f_2$ , namely  $2/3, 3/2$ , and  $(13 \pm \sqrt{601})/12$ .

**Remark:** The fact that the two conics in  $\mathbb{P}^2(\mathbb{C})$  meet in four points, counted with multiplicities, is a special case of *Bézout's theorem*: two curves in  $\mathbb{P}^2(\mathbb{C})$  of degrees  $m, n$  and not sharing any common component meet in exactly  $mn$  points when counted with multiplicity.

Many solvers were surprised that the proposers chose the parameter  $1/24$  to give two rational roots and two nonrational roots. In fact, they had no choice in the matter: attempting to make all four roots rational by replacing  $1/24$  by  $\beta$  amounts to asking for  $\beta^2 + \beta$  and  $\beta^2 + \beta + 1$  to be perfect squares. This cannot happen outside of trivial cases ( $\beta = 0, -1$ ) ultimately because the elliptic curve 24A1 (in Cremona's notation) over  $\mathbb{Q}$  has rank 0. (Thanks to Noam Elkies for providing this computation.)

However, there are choices that make the radical milder, e.g.,  $\beta = 1/3$  gives  $\beta^2 + \beta = 4/9$  and  $\beta^2 + \beta + 1 = 13/9$ , while  $\beta = 3/5$  gives  $\beta^2 + \beta = 24/25$  and  $\beta^2 + \beta + 1 = 49/25$ .

A-2 The minimum is 4, achieved by the square with vertices  $(\pm 1, \pm 1)$ .

**First solution:** To prove that 4 is a lower bound, let  $S$  be a convex set of the desired form. Choose  $A, B, C, D \in S$  lying on the branches of the two hyperbolas, with  $A$  in the upper right quadrant,  $B$  in the upper left,  $C$  in the lower left,  $D$  in the lower right. Then the area of the quadrilateral  $ABCD$  is a lower bound for the area of  $S$ .

Write  $A = (a, 1/a)$ ,  $B = (-b, 1/b)$ ,  $C = (-c, -1/c)$ ,  $D = (d, -1/d)$  with  $a, b, c, d > 0$ . Then the area of the quadrilateral  $ABCD$  is

$$\frac{1}{2}(a/b + b/c + c/d + d/a + b/a + c/b + d/c + a/d),$$

which by the arithmetic-geometric mean inequality is at least 4.

**Second solution:** Choose  $A, B, C, D$  as in the first solution. Note that both the hyperbolas and the area of the convex hull of  $ABCD$  are invariant under the transformation  $(x, y) \mapsto (xm, y/m)$  for any  $m > 0$ . For  $m$  small, the counterclockwise angle from the line  $AC$  to the line  $BD$  approaches 0; for  $m$  large, this angle approaches  $\pi$ . By continuity, for some  $m$  this angle becomes  $\pi/2$ , that is,  $AC$  and  $BD$  become perpendicular. The area of  $ABCD$  is then  $AC \cdot BD$ .

It thus suffices to note that  $AC \geq 2\sqrt{2}$  (and similarly for  $BD$ ). This holds because if we draw the tangent lines to the hyperbola  $xy = 1$  at the points  $(1, 1)$  and  $(-1, -1)$ , then  $A$  and  $C$  lie outside the region between these lines. If we project the segment  $AC$  orthogonally onto the line  $x = y = 1$ , the resulting projection has length at least  $2\sqrt{2}$ , so  $AC$  must as well.

**Third solution:** (by Richard Stanley) Choose  $A, B, C, D$  as in the first solution. Now fixing  $A$  and  $C$ , move  $B$  and  $D$  to the points at which the tangents to the curve are parallel to the line  $AC$ . This does not increase the area of the quadrilateral  $ABCD$  (even if this quadrilateral is not convex).

Note that  $B$  and  $D$  are now diametrically opposite; write  $B = (-x, 1/x)$  and  $D = (x, -1/x)$ . If we thus repeat the procedure, fixing  $B$  and  $D$  and moving  $A$  and  $C$  to the points where the tangents are parallel to  $BD$ , then  $A$  and  $C$  must move to  $(x, 1/x)$  and  $(-x, -1/x)$ , respectively, forming a rectangle of area 4.

**Remark:** Many geometric solutions are possible. An example suggested by David Savitt (due to Chris Brewer): note that  $AD$  and  $BC$  cross the positive and negative  $x$ -axes, respectively, so the convex hull of  $ABCD$  contains  $O$ . Then check that the area of triangle  $OAB$  is at least 1, et cetera.

A-3 Assume that we have an ordering of  $1, 2, \dots, 3k+1$  such that no initial subsequence sums to  $0 \pmod{3}$ . If we omit the multiples of 3 from this ordering, then the remaining sequence mod 3 must look like  $1, 1, -1, 1, -1, \dots$  or  $-1, -1, 1, -1, 1, \dots$ . Since there is one more integer in the ordering congruent to  $1 \pmod{3}$  than to  $-1$ , the sequence mod 3 must look like  $1, 1, -1, 1, -1, \dots$

It follows that the ordering satisfies the given condition if and only if the following two conditions hold: the first element in the ordering is not divisible by 3, and the sequence mod 3 (ignoring zeroes) is of the form  $1, 1, -1, 1, -1, \dots$ . The two conditions are independent, and the probability of the first is  $(2k+1)/(3k+1)$  while the probability of the second is  $1/\binom{2k+1}{k}$ , since there are  $\binom{2k+1}{k}$  ways to order  $(k+1)$  1's and  $k$  -1's. Hence the desired probability is the product of these two, or  $\frac{k!(k+1)!}{(3k+1)(2k)!}$ .

A-4 Note that  $n$  is a repunit if and only if  $9n+1 = 10^m$  for some power of 10 greater than 1. Consequently, if we put

$$g(n) = 9f\left(\frac{n-1}{9}\right) + 1,$$

then  $f$  takes repunits to repunits if and only if  $g$  takes powers of 10 greater than 1 to powers of 10 greater than 1. We will show that the only such functions  $g$  are those of the form  $g(n) = 10^c n^d$  for  $d \geq 0$ ,  $c \geq 1-d$  (all of which clearly work), which will mean that the desired polynomials  $f$  are those of the form

$$f(n) = \frac{1}{9}(10^c(9n+1)^d - 1)$$

for the same  $c, d$ .

It is convenient to allow "powers of 10" to be of the form  $10^k$  for any integer  $k$ . With this convention, it suffices to check that the polynomials  $g$  taking powers of 10 greater than 1 to powers of 10 are of the form  $10^c n^d$  for any integers  $c, d$  with  $d \geq 0$ .

**First solution:** Suppose that the leading term of  $g(x)$  is  $ax^d$ , and note that  $a > 0$ . As  $x \rightarrow \infty$ , we have  $g(x)/x^d \rightarrow a$ ; however, for  $x$  a power of 10 greater than 1,  $g(x)/x^d$  is a power of 10. The set of powers of 10 has no positive limit point, so  $g(x)/x^d$  must be equal to  $a$  for  $x = 10^k$  with  $k$  sufficiently large, and we must have  $a = 10^c$  for some  $c$ . The polynomial  $g(x) - 10^c x^d$  has infinitely many roots, so must be identically zero.

**Second solution:** We proceed by induction on  $d = \deg(g)$ . If  $d = 0$ , we have  $g(n) = 10^c$  for some  $c$ . Otherwise,  $g$  has rational coefficients by Lagrange's interpolation formula (this applies to any polynomial of degree  $d$  taking at least  $d+1$  different rational numbers to rational numbers), so  $g(0) = t$  is rational. Moreover,  $g$  takes each value only finitely many times, so the sequence  $g(10^0), g(10^1), \dots$  includes arbitrarily large powers of 10. Suppose that  $t \neq 0$ ; then we can choose a positive integer  $h$  such that the numerator of  $t$  is not divisible by  $10^h$ . But for  $c$  large enough,  $g(10^c) - t$  has numerator divisible by  $10^h$  for some  $b > h$ , contradiction.

Consequently,  $t = 0$ , and we may apply the induction hypothesis to  $g(n)/n$  to deduce the claim.

**Remark:** The second solution amounts to the fact that  $g$ , being a polynomial with rational coefficients, is continuous for the 2-adic and 5-adic topologies on  $\mathbb{Q}$ . By contrast, the first solution uses the “ $\infty$ -adic” topology, i.e., the usual real topology.

A–5 In all solutions, let  $G$  be a finite group of order  $m$ .

**First solution:** By Lagrange’s theorem, if  $m$  is not divisible by  $p$ , then  $n = 0$ . Otherwise, let  $S$  be the set of  $p$ -tuples  $(a_0, \dots, a_{p-1}) \in G^p$  such that  $a_0 \cdots a_{p-1} = e$ ; then  $S$  has cardinality  $m^{p-1}$ , which is divisible by  $p$ . Note that this set is invariant under cyclic permutation, that is, if  $(a_0, \dots, a_{p-1}) \in S$ , then  $(a_1, \dots, a_{p-1}, a_0) \in S$  also. The fixed points under this operation are the tuples  $(a, \dots, a)$  with  $a^p = e$ ; all other tuples can be grouped into orbits under cyclic permutation, each of which has size  $p$ . Consequently, the number of  $a \in G$  with  $a^p = e$  is divisible by  $p$ ; since that number is  $n + 1$  (only  $e$  has order 1), this proves the claim.

**Second solution:** (by Anand Deopurkar) Assume that  $n > 0$ , and let  $H$  be any subgroup of  $G$  of order  $p$ . Let  $S$  be the set of all elements of  $G \setminus H$  of order dividing  $p$ , and let  $H$  act on  $G$  by conjugation. Each orbit has size  $p$  except for those which consist of individual elements  $g$  which commute with  $H$ . For each such  $g$ ,  $g$  and  $H$  generate an elementary abelian subgroup of  $G$  of order  $p^2$ . However, we can group these  $g$  into sets of size  $p^2 - p$  based on which subgroup they generate together with  $H$ . Hence the cardinality of  $S$  is divisible by  $p$ ; adding the  $p - 1$  nontrivial elements of  $H$  gives  $n \equiv -1 \pmod{p}$  as desired.

**Third solution:** Let  $S$  be the set of elements in  $G$  having order dividing  $p$ , and let  $H$  be an elementary abelian  $p$ -group of maximal order in  $G$ . If  $|H| = 1$ , then we are done. So assume  $|H| = p^k$  for some  $k \geq 1$ , and let  $H$  act on  $S$  by conjugation. Let  $T \subset S$  denote the set of fixed points of this action. Then the size of every  $H$ -orbit on  $S$  divides  $p^k$ , and so  $|S| \equiv |T| \pmod{p}$ . On the other hand,  $H \subset T$ , and if  $T$  contained an element not in  $H$ , then that would contradict the maximality of  $H$ . It follows that  $H = T$ , and so  $|S| \equiv |T| = |H| = p^k \equiv 0 \pmod{p}$ , i.e.,  $|S| = n + 1$  is a multiple of  $p$ .

**Remark:** This result is a theorem of Cauchy; the first solution above is due to McKay. A more general (and more difficult) result was proved by Frobenius: for any positive integer  $m$ , if  $G$  is a finite group of order divisible by  $m$ , then the number of elements of  $G$  of order dividing  $m$  is a multiple of  $m$ .

A–6 For an admissible triangulation  $\mathcal{T}$ , number the vertices of  $P$  consecutively  $v_1, \dots, v_n$ , and let  $a_i$  be the number of edges in  $\mathcal{T}$  emanating from  $v_i$ ; note that  $a_i \geq 2$  for all  $i$ .

We first claim that  $a_1 + \dots + a_n \leq 4n - 6$ . Let  $V, E, F$  denote the number of vertices, edges, and faces in  $\mathcal{T}$ . By Euler’s Formula,  $(F + 1) - E + V = 2$  (one must add 1 to the face count for the region exterior to  $P$ ). Each

face has three edges, and each edge but the  $n$  outside edges belongs to two faces; hence  $F = 2E - n$ . On the other hand, each edge has two endpoints, and each of the  $V - n$  internal vertices is an endpoint of at least 6 edges; hence  $a_1 + \dots + a_n + 6(V - n) \leq 2E$ . Combining this inequality with the previous two equations gives

$$\begin{aligned} a_1 + \dots + a_n &\leq 2E + 6n - 6(1 - F + E) \\ &= 4n - 6, \end{aligned}$$

as claimed.

Now set  $A_3 = 1$  and  $A_n = A_{n-1} + 2n - 3$  for  $n \geq 4$ ; we will prove by induction on  $n$  that  $\mathcal{T}$  has at most  $A_n$  triangles. For  $n = 3$ , since  $a_1 + a_2 + a_3 = 6$ ,  $a_1 = a_2 = a_3 = 2$  and hence  $\mathcal{T}$  consists of just one triangle.

Next assume that an admissible triangulation of an  $(n - 1)$ -gon has at most  $A_{n-1}$  triangles, and let  $\mathcal{T}$  be an admissible triangulation of an  $n$ -gon. If any  $a_i = 2$ , then we can remove the triangle of  $\mathcal{T}$  containing vertex  $v_i$  to obtain an admissible triangulation of an  $(n - 1)$ -gon; then the number of triangles in  $\mathcal{T}$  is at most  $A_{n-1} + 1 < A_n$  by induction. Otherwise, all  $a_i \geq 3$ . Now the average of  $a_1, \dots, a_n$  is less than 4, and thus there are more  $a_i = 3$  than  $a_i \geq 5$ . It follows that there is a sequence of  $k$  consecutive vertices in  $P$  whose degrees are  $3, 4, 4, \dots, 4, 3$  in order, for some  $k$  with  $2 \leq k \leq n - 1$  (possibly  $k = 2$ , in which case there are no degree 4 vertices separating the degree 3 vertices). If we remove from  $\mathcal{T}$  the  $2k - 1$  triangles which contain at least one of these vertices, then we are left with an admissible triangulation of an  $(n - 1)$ -gon. It follows that there are at most  $A_{n-1} + 2k - 1 \leq A_{n-1} + 2n - 3 = A_n$  triangles in  $\mathcal{T}$ . This completes the induction step and the proof.

**Remark:** We can refine the bound  $A_n$  somewhat. Supposing that  $a_i \geq 3$  for all  $i$ , the fact that  $a_1 + \dots + a_n \leq 4n - 6$  implies that there are at least six more indices  $i$  with  $a_i = 3$  than with  $a_i \geq 5$ . Thus there exist six sequences with degrees  $3, 4, \dots, 4, 3$ , of total length at most  $n + 6$ . We may thus choose a sequence of length  $k \leq \lfloor \frac{n}{6} \rfloor + 1$ , so we may improve the upper bound to  $A_n = A_{n-1} + 2\lfloor \frac{n}{6} \rfloor + 1$ , or asymptotically  $\frac{1}{6}n^2$ .

However (as noted by Noam Elkies), a hexagonal swatch of a triangular lattice, with the boundary as close to regular as possible, achieves asymptotically  $\frac{1}{6}n^2$  triangles.

B–1 The problem fails if  $f$  is allowed to be constant, e.g., take  $f(n) = 1$ . We thus assume that  $f$  is nonconstant. Write  $f(n) = \sum_{i=0}^d a_i n^i$  with  $a_i > 0$ . Then

$$\begin{aligned} f(f(n) + 1) &= \sum_{i=0}^d a_i (f(n) + 1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If  $n = 1$ , then this implies that  $f(f(n) + 1)$  is divisible by  $f(n)$ . Otherwise,  $0 < f(1) < f(n)$  since  $f$  is nonconstant and has positive coefficients, so  $f(f(n) + 1)$  cannot be divisible by  $f(n)$ .

B-2 Put  $B = \max_{0 \leq x \leq 1} |f'(x)|$  and  $g(x) = \int_0^x f(y) dy$ . Since  $g(0) = g(1) = 0$ , the maximum value of  $|g(x)|$  must occur at a critical point  $y \in (0, 1)$  satisfying  $g'(y) = f(y) = 0$ . We may thus take  $\alpha = y$  hereafter.

Since  $\int_0^\alpha f(x) dx = -\int_0^{1-\alpha} f(1-x) dx$ , we may assume that  $\alpha \leq 1/2$ . By then substituting  $-f(x)$  for  $f(x)$  if needed, we may assume that  $\int_0^\alpha f(x) dx \geq 0$ . From the inequality  $f'(x) \geq -B$ , we deduce  $f(x) \leq B(\alpha - x)$  for  $0 \leq x \leq \alpha$ , so

$$\begin{aligned} \int_0^\alpha f(x) dx &\leq \int_0^\alpha B(\alpha - x) dx \\ &= -\frac{1}{2}B(\alpha - x)^2 \Big|_0^\alpha \\ &= \frac{\alpha^2}{2}B \leq \frac{1}{8}B \end{aligned}$$

as desired.

B-3 **First solution:** Observing that  $x_2/2 = 13$ ,  $x_3/4 = 34$ ,  $x_4/8 = 89$ , we guess that  $x_n = 2^{n-1}F_{2n+3}$ , where  $F_k$  is the  $k$ -th Fibonacci number. Thus we claim that  $x_n = \frac{2^{n-1}}{\sqrt{5}}(\alpha^{2n+3} + \alpha^{-(2n+3)})$ , where  $\alpha = \frac{1+\sqrt{5}}{2}$ , to make the answer  $x_{2007} = \frac{2^{2006}}{\sqrt{5}}(\alpha^{4017} + \alpha^{-4017})$ .

We prove the claim by induction; the base case  $x_0 = 1$  is true, and so it suffices to show that the recursion  $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$  is satisfied for our formula for  $x_n$ . Indeed, since  $\alpha^2 = \frac{3+\sqrt{5}}{2}$ , we have

$$\begin{aligned} x_{n+1} - (3 + \sqrt{5})x_n &= \frac{2^{n-1}}{\sqrt{5}}(2(\alpha^{2n+5} + \alpha^{-(2n+5)}) \\ &\quad - (3 + \sqrt{5})(\alpha^{2n+3} + \alpha^{-(2n+3)})) \\ &= -2^n \alpha^{-(2n+3)}. \end{aligned}$$

Now  $2^n \alpha^{-(2n+3)} = (\frac{\sqrt{5}-1}{2})^3 (3 - \sqrt{5})^n$  is between 0 and 1; the recursion follows since  $x_n, x_{n+1}$  are integers.

**Second solution:** (by Catalin Zara) Since  $x_n$  is rational, we have  $0 < x_n \sqrt{5} - \lfloor x_n \sqrt{5} \rfloor < 1$ . We now have the inequalities

$$\begin{aligned} x_{n+1} - 3x_n &< x_n \sqrt{5} < x_{n+1} - 3x_n + 1 \\ (3 + \sqrt{5})x_n - 1 &< x_{n+1} < (3 + \sqrt{5})x_n \\ 4x_n - (3 - \sqrt{5}) &< (3 - \sqrt{5})x_{n+1} < 4x_n \\ 3x_{n+1} - 4x_n &< x_{n+1} \sqrt{5} < 3x_{n+1} - 4x_n + (3 - \sqrt{5}). \end{aligned}$$

Since  $0 < 3 - \sqrt{5} < 1$ , this yields  $\lfloor x_{n+1} \sqrt{5} \rfloor = 3x_{n+1} - 4x_n$ , so we can rewrite the recursion as  $x_{n+1} = 6x_n - 4x_{n-1}$  for  $n \geq 2$ . It is routine to solve this recursion to obtain the same solution as above.

**Remark:** With an initial 1 prepended, this becomes sequence A018903 in Sloane's On-Line Encyclopedia of Integer Sequences: (<http://www.research.att.com/~njas/>

sequences/). Therein, the sequence is described as the case  $S(1, 5)$  of the sequence  $S(a_0, a_1)$  in which  $a_{n+2}$  is the least integer for which  $a_{n+2}/a_{n+1} > a_{n+1}/a_n$ . Sloane cites D. W. Boyd, Linear recurrence relations for some generalized Pisot sequences, *Advances in Number Theory* (Kingston, ON, 1991), Oxford Univ. Press, New York, 1993, p. 333–340.

B-4 The number of pairs is  $2^{n+1}$ . The degree condition forces  $P$  to have degree  $n$  and leading coefficient  $\pm 1$ ; we may count pairs in which  $P$  has leading coefficient 1 as long as we multiply by 2 afterward.

Factor both sides:

$$\begin{aligned} (P(X) + Q(X)i)(P(X) - Q(X)i) \\ &= \prod_{j=0}^{n-1} (X - \exp(2\pi i(2j+1)/(4n))) \\ &\quad \cdot \prod_{j=0}^{n-1} (X + \exp(2\pi i(2j+1)/(4n))). \end{aligned}$$

Then each choice of  $P, Q$  corresponds to equating  $P(X) + Q(X)i$  with the product of some  $n$  factors on the right, in which we choose exactly of the two factors for each  $j = 0, \dots, n-1$ . (We must take exactly  $n$  factors because as a polynomial in  $X$  with complex coefficients,  $P(X) + Q(X)i$  has degree exactly  $n$ . We must choose one for each  $j$  to ensure that  $P(X) + Q(X)i$  and  $P(X) - Q(X)i$  are complex conjugates, so that  $P, Q$  have real coefficients.) Thus there are  $2^n$  such pairs; multiplying by 2 to allow  $P$  to have leading coefficient  $-1$  yields the desired result.

**Remark:** If we allow  $P$  and  $Q$  to have complex coefficients but still require  $\deg(P) > \deg(Q)$ , then the number of pairs increases to  $2 \binom{2n}{n}$ , as we may choose any  $n$  of the  $2n$  factors of  $X^{2n} + 1$  to use to form  $P(X) + Q(X)i$ .

B-5 For  $n$  an integer, we have  $\lfloor \frac{n}{k} \rfloor = \frac{n-j}{k}$  for  $j$  the unique integer in  $\{0, \dots, k-1\}$  congruent to  $n$  modulo  $k$ ; hence

$$\prod_{j=0}^{k-1} \left( \lfloor \frac{n}{k} \rfloor - \frac{n-j}{k} \right) = 0.$$

By expanding this out, we obtain the desired polynomials  $P_0(n), \dots, P_{k-1}(n)$ .

**Remark:** Variants of this solution are possible that construct the  $P_i$  less explicitly, using Lagrange interpolation or Vandermonde determinants.

B-6 (Suggested by Oleg Golberg) Assume  $n \geq 2$ , or else the problem is trivially false. Throughout this proof, any  $C_i$  will be a positive constant whose exact value is immaterial. As in the proof of Stirling's approximation, we estimate for any fixed  $c \in \mathbb{R}$ ,

$$\sum_{i=1}^n (i+c) \log i = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + O(n \log n)$$

by comparing the sum to an integral. This gives

$$\begin{aligned} n^{n^2/2-C_1n} e^{-n^2/4} &\leq 1^{1+c} 2^{2+c} \dots n^{n+c} \\ &\leq n^{n^2/2+C_2n} e^{-n^2/4}. \end{aligned}$$

We now interpret  $f(n)$  as counting the number of  $n$ -tuples  $(a_1, \dots, a_n)$  of nonnegative integers such that

$$a_1 1! + \dots + a_n n! = n!.$$

For an upper bound on  $f(n)$ , we use the inequalities  $0 \leq a_i \leq n!/i!$  to deduce that there are at most  $n!/i! +$

$1 \leq 2(n!/i!)$  choices for  $a_i$ . Hence

$$\begin{aligned} f(n) &\leq 2^n \frac{n!}{1!} \dots \frac{n!}{n!} \\ &= 2^n 2^1 3^2 \dots n^{n-1} \\ &\leq n^{n^2/2+C_3n} e^{-n^2/4}. \end{aligned}$$

For a lower bound on  $f(n)$ , we note that if  $0 \leq a_i < (n-1)!/i!$  for  $i = 2, \dots, n-1$  and  $a_n = 0$ , then  $0 \leq a_2 2! + \dots + a_n n! \leq n!$ , so there is a unique choice of  $a_1$  to complete this to a solution of  $a_1 1! + \dots + a_n n! = n!$ . Hence

$$\begin{aligned} f(n) &\geq \frac{(n-1)!}{2!} \dots \frac{(n-1)!}{(n-1)!} \\ &= 3^1 4^2 \dots (n-1)^{n-3} \\ &\geq n^{n^2/2+C_4n} e^{-n^2/4}. \end{aligned}$$