1. Let S be a set, and (A_1, B_1) and (A_2, B_2) be two partitions of S. Hence for $i \in \{1, 2\}$, A_i and B_i are disjoint, and $A_i \cup B_i = S$. Which of the following statements about the sets A_1, B_1, A_2, and B_2 MUST be true?

(a) $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \emptyset$. ✓

(b) $(A_1 \cup A_2) \cap (B_1 \cup B_2) = \emptyset$.

(c) $(A_1 \cap A_2) \cup (B_1 \cap B_2) = S$.

(d) $(A_1 \cap A_2) \cup (B_1 \cap B_2) = \emptyset$.

(e) None of the above.

2. Let $f : S \to S$ be any function. For $X, Y \subseteq S$, define

$$f(X) = \{\, f(x) : x \in X \,\}, \qquad f^{-1}(Y) = \{\, y \in S : \; f(y) \in Y \,\}.$$

Which of the following statements holds for *every* function f and all subsets A, B?

(a) $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq S$.

(b) $f^{-1}\big(f(A)\big) = A$ for all $A \subseteq S$.

(c) $f(A \setminus B) = f(A) \setminus f(B)$ for all $A, B \subseteq S$.

(d) If f is surjective, then $f(S \setminus A) = S \setminus f(A)$ for all $A \subseteq S$.

(e) $f\big(f^{-1}(B)\big) = B \cap f(S)$ **for all** $B \subseteq S$**.** ✓

3. How many *surjective* functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ are there?

(a) 4^5

(b) $4!$

(c) $4^5 - 3^5 + 2^5 - 1^5$

(d) $4^5 - 4 \times 3^5 + 6 \times 2^5 - 4 \times 1^5$ ✓

(e) $\dfrac{5!}{4!}$

4. "FizzBuzz" sequence starts with the integer 1, keeps incrementing by 1, but replacing any number divisible by 3 by 'Fizz', any number divisible by 5 by 'Buzz', and any number divisible by both 3 and 5 by 'FizzBuzz'. The first few terms in the sequence is of the form:

1, 2, Fizz, 4, Buzz, Fizz, 7, 8, Fizz, Buzz, 11, Fizz, 13, 14, FizzBuzz, 16, $\cdots$

The 8th number that appears in the above sequence is 14. Which of the following is the 100th number?

(a) 151

(b) 173

(c) 178

(d) 179

(e) 187 ✓

5. Consider the set of all distinct 6-letter sequences obtained by shuffling the letters of the word ‘banana’. If you arrange this list in alphabetical order (with the first sequence in this list being ’aaabnn’ and the last sequence in this list being ’nnbaaa’), what is the 42nd sequence in this list?

(a) bannaa

(b) bnaaan

(c) bnaana

(d) naaanb ✓

(e) nabaan

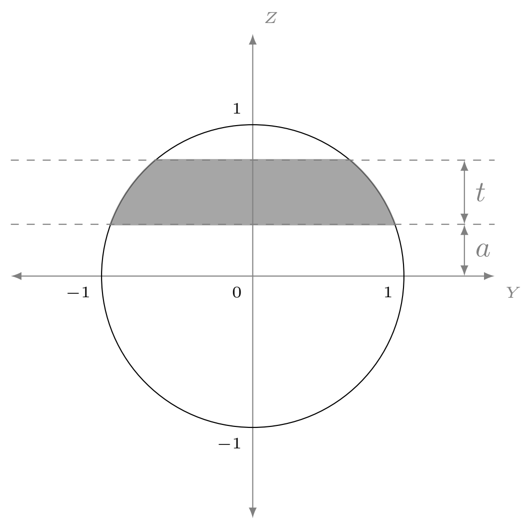
6. Let $\mathbb{S} := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. For $0 \leq a \leq b \leq 1$, denote by $A(a, b)$ the surface area of the subset

$$\{(x, y, z) \in \mathbb{S} | a \leq z \leq b\} .$$

For $t \in (0, 1)$, consider the function $f_t : [0, 1 - t] \to \mathbb{R}$ defined by

$$f_t(a) := A(a, a + t).$$

We emphasize that for each $t \in (0, 1)$, the domain of f_t is the closed interval $[0, 1-t]$. The shaded part in the figure below shows the subset above whose area is $A(a, a+t)$, as seen when viewing the sphere from a direction perpendicular to the YZ-plane.



Consider the following statements.

(i) There exists a $t \in (0, 1)$ for which f_t is strictly increasing on its domain.

(ii) There exists a $t \in (0, 1)$ for which f_t is strictly decreasing on its domain.

(iii) There exists a $t \in (0, 1)$ for which f_t is constant on its domain.

Which of the above statements is/are true?

(a) (i) only

(b) (ii) only

(c) (iii) only ✓

(d) (i), (ii), and (iii) are all true.

(e) None of (i), (ii), and (iii) is true.

7. Consider the unit interval $I = [0, 1]$. A point P is chosen randomly from I according to a probability density function on I in which the density at any point $x \in I$ is proportional to the distance of x from the endpoint of I nearest to x. What is expected distance of the randomly chosen point P from the endpoint of I **nearest** to P?

(a) $1/2$

(b) $1/3$ ✓

(c) $1/4$

(d) $1/5$

(e) $1/6$

8. Two points are chosen independently and uniformly at random on the circumference of a circle of radius 1. What is the expected length of the chord joining the two chosen points?

(a) $6/\pi$

(b) $5/\pi$

(c) $4/\pi$ ✓

(d) $3/\pi$

(e) $2/\pi$

9. Two positive real numbers x and y are chosen independently and uniformly at random in $[0, 2]$. What is the probability that $xy < 1$ and $\frac{y}{x} < 2$?

(a) $\frac{3 \ln 2}{4}$

(b) $\frac{3 \ln 2 + 1}{4}$

(c) $\frac{3 \ln 2}{8}$

(d) $\frac{3 \ln 2 + 1}{8}$ ✓

(e) None of the above

10. Let $\mathbb{N}$ be the set of all positive integers and $a, b \in \mathbb{N}$ be such that their greatest common divisor is 1. Let

$$S(a, b) := \{a \cdot x + b \cdot y \mid x, y \in \mathbb{N}\}.$$

Let $T(a, b) := \mathbb{N} \setminus S(a, b)$. Consider the following statements.

(i) $T(a, b)$ is always a finite set.

(ii) $T(a, b)$ need not be a finite set.

(iii) The largest number in $T(a, b)$ is at least $a \cdot b$.

(iv) The largest number in $T(a, b)$ is at most $a \cdot b$.

Which of the above is/are true?

(a) Only (i)

(b) Only (ii)

(c) Only (ii) and (iii)

(d) Only (i) and (iv) ✓

(e) Only (ii) and (iv)

11. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Let $A \in M_n(\mathbb{R})$ be a diagonal matrix with n **distinct** entries on the diagonal. Consider the linear map

$$\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R}), \qquad \Phi(X) = AX - XA.$$

What is the dimension of the image of Φ?

(a) 0

(b) n

(c) $n(n-1)/2$

(d) $n^2 - n$ ✓

(e) n^2

12. Define $n = 20^{20} \cdot 25^{25}$. What is the number of factors of n^2 that are smaller than n?

(a) 500

(b) 1490

(c) 5710 ✓

(d) 4255

(e) 2800

13. Uma loves numbers that are multiples of 3, while Gauri does not. Anytime Uma sees an integer n that is not a multiple of 3, she is willing to pay ₹1 to round it down to the nearest multiple of 3 and anytime Gauri sees an integer n that is a multiple of 3, she is willing to pay ₹1 to divide the number by 3. The process stops when the number becomes 0. Given an integer n, let $u(n)$ and $g(n)$ be the money spent by Uma and Gauri during the process respectively. Consider the following statements for $0 \leq n < 3^{2025}$:

(i) $u(n) + g(n) > 2025$.

(ii) $u(n) > 2025$.

(iii) $u(n) > g(n) + 1$.

(iv) $g(n) > 2025$.

Which of the above are possible?

(a) None of (i), (ii), (iii), and (iv).

(b) Only (i) but not (ii), (iii), and (iv). ✓

(c) Only (i) and (ii) but not (iii) and (iv).

(d) Only (i), (ii), and (iii) but not (iv).

(e) All of (i), (ii), (iii), and (iv).

14. A 4x4 sudoku is a 4×4 grid partitioned into four 2×2 squares or "boxes". The following rules have to be obeyed.

- Rows: Each row is a permutation of 1, 2, 3, 4.
- Columns: Each column is a permutation of 1, 2, 3, 4.
- Boxes: Each box is a permutation of 1, 2, 3, 4.

An example of a sudoku grid is the following.

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

What is the total number of distinct sudoku grids?

(a) 288 ✓

(b) 384

(c) 576

(d) 1152

(e) None of the above

15. Let B_2^n denote the Euclidean unit ball and $C_n = [-1,1]^n$ the cube in $\mathbb{R}^n$. Which statement is true?

(a) $\mathrm{Vol}(B_2^n)$ grows with n.

(b) $\mathrm{Vol}(C_n)/\mathrm{Vol}(B_2^n)$ tends to 0.

(c) $\mathrm{Vol}(B_2^n)$ tends to 0 as n increases. ✓

(d) B_2^n and C_n have the same volume for all n.

(e) $\mathrm{Vol}(B_2^n)$ approaches ∞.

1. What is the decimal representation of the hexadecimal number $(\text{A}1.4\text{C})_{16}$?

(a) $11\frac{13}{256}$

(b) $17\frac{13}{256}$

(c) $161\frac{71}{256}$

(d) $161\frac{13}{256}$

(e) None of the above. ✓

2. Given an undirected graph $G = (V, E)$, the edge chromatic number of G, denoted by $\chi'(G)$, is the mimimum number of colours required to colour the edges of the graph G such that no two edges incident on the same vertex have the same colour. It is not hard to see that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of the graph. For which of the following graphs $\chi'(G) > \Delta(G)$?

(a) C_6, the cycle of length 6

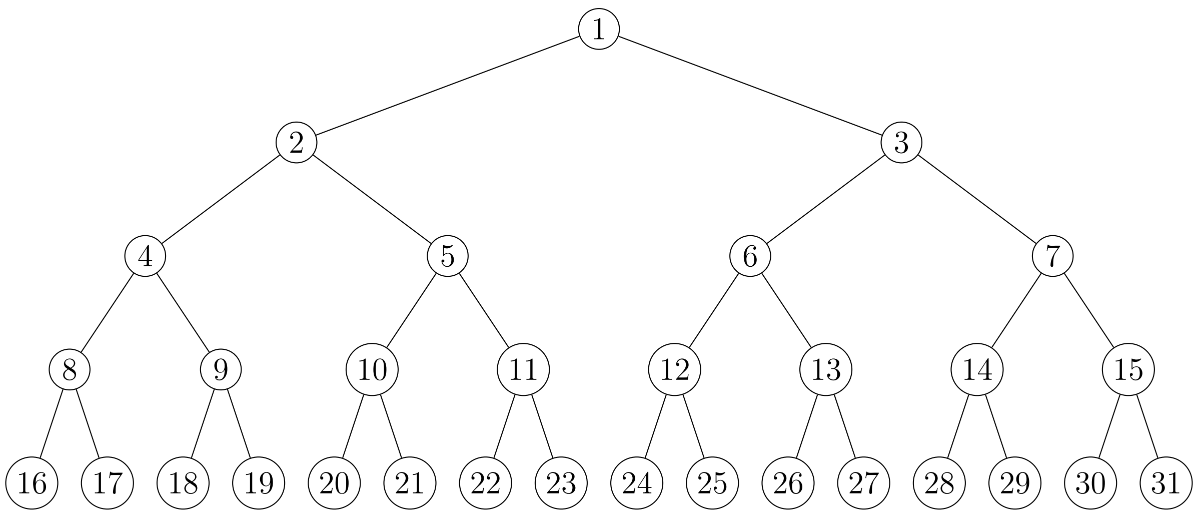
(b) C_5, the cycle of length 5 ✓

(c) K_4, the complete graph on 4 vertices

(d) $K_{2,2}$, the complete bipartite graph with 2 vertices on each side

(e) The balanced binary tree of depth 3

3. A coder claims to have implemented the depth first search algorithm correctly and a tester has to verify this claim. For this, the tester runs the implementation on an input graph G and the result is a complete binary tree of depth 4, like in the picture below.



The presence of which of the following edges in G would prove that the implementation is incorrect?

(a) $(1, 5)$

(b) $(2, 10)$

(c) $(3, 15)$

(d) $(4, 20)$ ✓

(e) None of the above

4. Let G be a weighted graph where the weight of an edge e is $w(e)$. Let S be a subset of the edges of G that form a minimum-weight spanning tree. Which of the following does **not** depend on the choice of S?

(i) $\sum_{e \in S\colon w(e) \text{ is prime}} w(e)$

(ii) $\sum_{e \in S\colon w(e) \text{ is irrational}} w(e)$

(iii) $\sum_{e \in S\colon w(e) < 2025} w(e)$

(a) (i) only

(b) (ii) only

(c) (iii) only

(d) More than one, but not all of (i), (ii), and (iii)

(e) All of (i)**,** (ii)**, and** (iii) ✓

5. For a language L, consider the language $\mathsf{perm}(L)$ defined as follows.

$$\mathsf{perm}(L) = \{w \mid \exists u \in L \text{ such that } u \text{ is a permutation of } w\}$$

Now consider the following statements.

(i) If L is regular, then $\mathsf{perm}(L)$ is also regular.

(ii) If L is context-free, then $\mathsf{perm}(L)$ is also context-free.

(iii) If L is decidable, then $\mathsf{perm}(L)$ is decidable.

Which of the above are true?

(a) Only (i) and (iii)

(b) Only (ii) and (iii)

(c) Only (iii) ✓

(d) All of (i), (ii), and (iii) are true

(e) None of (i), (ii), or (iii) is true

6. Consider an alphabet Σ. Let Σ^+ denote the set of non-empty words over Σ. Let $U, V \subseteq \Sigma^+$. Let U^ω denote the set of infinite words formed by concatenating words from U infinitely many times, that is,

$$U^\omega = \{w = u_1 u_2 u_3 \cdots \mid u_i \in U \text{ for all } i \geqslant 0\}.$$

Also let $\lim(U)$ be the set of infinite words that have infinitely many prefixes in U, that is,

$$\lim(U) = \{w = a_1 a_2 a_3 \cdots | a_j \in \Sigma \text{ for all } j \geqslant 1 \text{ and } a_1 \cdots a_i \in U \text{ for infinitely many } i \geqslant 0\}.$$

Now consider the following statements.

(i) $(U \cup V)^\omega = U^\omega \cup V^\omega$.

(ii) $\lim(U \cup V) = \lim(U) \cup \lim(V)$.

Which of the above are true?

(a) (i) is true but (ii) is false

(b) (i) is false but (ii) is true ✓

(c) Both (i) and (ii) are true

(d) Both (i) and (ii) are false

(e) Both (i) and (ii) are true if and only if Σ has at least five letters

7. There are n different bottles, each containing a distinct type of wine. We know that exactly one of them is poisoned. Additionally, we have access to test kits that can take a drop of wine from a set of bottles and detect whether the poisoned bottle is in the set. This detection takes one day. What is the minimum number of test kits needed to find out the poisoned bottle within a day?

(a) $\Theta(n)$

(b) $\Theta(\sqrt{n})$

(c) $\Theta(\log n)$ ✓

(d) $\Theta(\log \log n)$

(e) None of the above

8. Let $G = (V, E)$ be an undirected simple graph on n vertices that has two distinct *perfect* matchings M_1 and M_2. Let E_1 be the set of those edges in M_1 that are not contained in M_2. Similarly, let E_2 be the set of those edges in M_2 that are not contained in M_1. Consider the edge subgraph $H = (V, E_1 \cup E_2)$ of G. What is the maximum possible number of vertices in H whose degree in H is an odd number? (Assume that $n \geq 100$.)

(a) 0 ✓

(b) 2

(c) 50

(d) $n/2$

(e) n

9. Consider a real $m \times n$ matrix A whose entries are drawn from the set $\{0, 1\}$. Given subsets S and T of $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$ respectively, we denote by $A_{S,T}$ the $|S| \times |T|$ submatrix of A obtained by picking from A the rows whose indices are in S and the columns whose indices are in T. Such a submatrix is called a *rectangle*.

A rectangle is said to be *on* if all its entries are 1. A rectangle $A_{S,T}$ is said to *include* the (i, j) entry of A if $i \in S$ and $j \in T$. Rectangles $A_{S,T}$ and $A_{S',T'}$ are said to be *disjoint* is the sets $S \times T$ and $S' \times T'$ are disjoint.

Suppose that there exist 100 disjoint rectangles of A, all of which are "on", such that only those entries of A are 1 that are included in one of these rectangles. Let $r(A)$ denote the rank of A. Which of the following statements is true for all such A?

(Assume that m, n are both at least 1000.)

(a) $r(A) = 100$

(b) $r(A) \leq 100$**, and** $r(A) = 1$ **is possible** ✓

(c) $50 \leq r(A) \leq 100$, and $r(A) = 50$ is possible

(d) $100 \leq r(A) \leq 200$, and $r(A) > 100$ is possible

(e) All of the above statements are false

10. Consider the path graph $G_n = (V_n, E_n)$ on n vertices as shown below. So $V_n = \{1, \ldots, n\}$ and $E_n = \{(i, i+1) : 1 \leq i \leq n-1\}$.

A matching M in G_n is a subset of E_n that contains at most one edge incident to each vertex. Note that the empty set is also a matching. So G_1 has one matching, $M = \emptyset$, while G_2 has two matchings, $M = \emptyset$ and $M = \{(1, 2)\}$.

How many distinct matchings are there in G_{10}?

(a) 89 ✓

(b) 100

(c) 512

(d) 1024

(e) None of the above

11. Consider two polynomials $p(x), q(x) \in \mathbb{Z}[x]$ of degree at most n, with each coefficient in $\{-1, 0, 1\}$. For which *smallest* positive integer M is it guaranteed that if $p(M) = q(M)$ then $p = q$?

(a) 2

(b) 3 ✓

(c) n

(d) $n!$

(e) there is no such M that works

12. Consider the recurrence

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) - T(1), \qquad T(1) = 1.$$

Here n is a positive integer. What is the asymptotic growth of $T(n)$?

(a) $\Theta(1)$ ✓

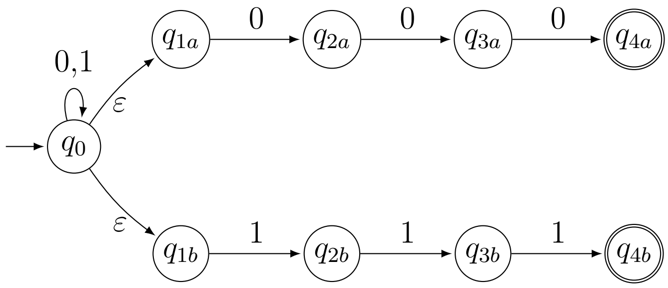
(b) $\Theta(\log n)$

(c) $\Theta(\sqrt{n})$

(d) $\Theta(n)$

(e) $\Theta(n \log n)$

13. Consider the following non-deterministic automaton.



Let p be the probability that a uniformly random string from $\{0,1\}^{10}$ is accepted by the above automaton. Which of the following is true about p?

(a) $0 < p \leq 0.05$

(b) $0.05 < p \leq 0.10$

(c) $0.10 < p \leq 0.15$

(d) $0.15 < p \leq 0.20$

(e) $p > 0.20$ ✓

14. Consider the following pseudocode that, on input a non-negative integer n, computes the remainder of the n-th Fibonacci number when divided by 100.

```
function Fib(n)
    if n = 0 then
        return 0
    else if n = 1 then
        return 1
    else
        a ←Fib(n − 1)
        b ←Fib(n − 2)
        return (a + b) mod 100
    end if
end function
```

In the above pseudocode, a and b are variables local to the function Fib.

Define $T : \mathbb{N} \to \mathbb{N}$ and $S : \mathbb{N} \to \mathbb{N}$ as follows: $T(n)$ and $S(n)$ is the time and space, respectively, taken by the above pseudocode to compute $\text{Fib}(n)$. Assume that recursion is implemented using the standard stack model.

If

$$\text{PolyBounded} = \{f : \mathbb{N} \to \mathbb{N} \ : \ \exists c > 0 \text{ such that } f(n) = O(n^c)\}$$
$$\text{Exp} = \{f : \mathbb{N} \to \mathbb{N} \ : \ f(n) = 2^{\Omega(n)}\}$$

then which of the following statements is true about $T(n)$ and $S(n)$?

(a) $T(n) \in$ PolyBounded and $S(n) \in$ PolyBounded.

(b) $T(n) \in$ PolyBounded and $S(n) \in$ Exp

(c) $T(n) \in$ Exp **and** $S(n) \in$ PolyBounded ✓

(d) $T(n) \in$ Exp and $S(n) \in$ Exp

(e) $T(n) \in$ Exp and $S(n) \notin$ Exp and $S(n) \notin$ PolyBounded

15. Let $f(x)$ be a non-zero univariate polynomial of degree $d > 1$ with integer coefficients. Over any ring or a field R that contains the ring of integers, we say that f is irreducible over R if f cannot be written as the product of two polynomials $g(x)$ and $h(x)$ over R such that each of g, h has degree at least one and has coefficients in the ring/field R.

Which of the following statements is true?

(a) f is irreducible over the ring of integers if and only if f is irreducible over the field of rational numbers. ✓

(b) f is irreducible over the ring of integers if and only if f is irreducible over the field of real numbers.

(c) f is irreducible over the field of rational numbers if and only if f is irreducible over the field of real numbers.

(d) f is irreducible over the field of rational numbers if and only if f is irreducible over the field of complex numbers.

(e) f is irreducible over the field of real numbers if and only if f is irreducible over the field of complex numbers.

1. Let B be the three dimensional Euclidean ball of radius 1 and let Q be the three dimensional cube $[-1,1]^3$. Let $B+Q$ be set of all vectors $x+y$ where $x \in B$ and $y \in Q$. Which of the following is FALSE?

(a) $B+Q$ is a convex set

(b) The volume of $B+Q$ is more than the sum of the volumes of B and Q

(c) $B+Q$ is not a cube

(d) The volume of $B+Q$ is $4\pi/3+24$ ✓

(e) $B+Q$ is not a Euclidean ball

2. Consider a uniform random point in the cube $[-1,1]^n$. As n increases, which statement is true?

(a) The average squared distance from the origin grows linearly with n ✓

(b) The average squared distance remains bounded

(c) The expected distance to the origin tends to 0

(d) The expectation of $\|x\|^2$ decreases as n increases

(e) The expectation of $\|x\|^3$ decreases as n increases

3. Let X and Y be independent standard normal random variables, i.e., independent Gaussian random variables with mean 0 and variance 1. Define the 2×2 random matrix

$$M = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}.$$

Let λ_1 and λ_2 be the two eigenvalues of M. Which of the following statements about (λ_1, λ_2) is **FALSE**?

(a) $\mathbb{E}[\lambda_1 + \lambda_2] = 0$

(b) $\mathrm{Cov}(\lambda_1, \lambda_2) = 0$

(c) The eigenvalues are $\lambda_1 = X+Y$ and $\lambda_2 = X-Y$

(d) λ_1 and λ_2 are independent random variables

(e) None of the above ✓

4. Consider a 2×2 symmetric matrix A with eigenvalues 5 and 2 corresponding to orthogonal eigenvectors $\vec{v}_1 = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}^T$ and $\vec{v}_2 = \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T$ respectively. That is,

$$A = 5\vec{v}_1\vec{v}_1^T + 2\vec{v}_2\vec{v}_2^T.$$

Consider the function $f(\vec{x}) = \vec{x}^T A \vec{x}$, where $\vec{x} \in \mathbb{R}^2$, and the set

$$S = \left\{ \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1 \text{ and } 2x_1 + x_2 = 0 \right\}.$$

What is the maximum value of $f(\vec{x})$ for $\vec{x} \in S$?

(a) 5

(b) 2 ✓

(c) 0

(d) 3

(e) None of the above

5. Consider a sequence of independent coin tosses, where each toss shows heads with probability p and tails with probability $1-p$. A "run" is a maximal sequence of consecutive identical outcomes, e.g., in HHTTTHTHH, the runs are HH, TTT, H, T, and HH.

Let R_n denote the number of runs in the first n tosses. For large n, what is the expected number of runs per toss, that is, $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[R_n]$?

(a) $p(1-p)$

(b) $(p(1-p))^2$

(c) $2p(1-p)$ ✓

(d) None of the above, but the limit exists

(e) The limit does not exist

6. Consider a particle starting at 0 on the integer number line. At each time step, it moves according to the following rule: with probability $1/2$, it moves right by 1 unit; and with probability $1/2$, it moves left by 1 unit. The particle stops moving when it first reaches either position $+3$ or position -2.

What is the probability that the particle reaches $+3$ before it reaches -2?

(a) $1/3$

(b) 2/5 ✓

(c) $1/2$

(d) $3/5$

(e) $2/3$

7. Suppose X is a uniform random variable which takes values in the interval $[0,1]$. We write this as $X \sim \mathsf{U}[0,1]$. Independent of the realization of X, with probability $p = 0.1$, an adversary is able to intercept X. On intercepting X, the adversary may replace it with a random value drawn according to a distribution of its choosing which may depend on the realization of X. Let Y denote the outcome after the possible intervention by the adversary. i.e., $Y = X$ with probability $1-p = 0.9$ and, Y is set by the adversary (possibly dependent on X) with probability $p = 0.1$. Consider the following distributions for Y:

(i) $Y \sim \mathsf{U}[0,1]$

(ii) $Y \sim \mathsf{U}[0.1, 1.1]$

(iii) $Y \sim \mathsf{U}[0, 1.1]$

Which of the above is a distribution of Y the adversary **CANNOT** induce?

(a) Only (i)

(b) Only (ii) ✓

(c) Only (iii)

(d) Only (ii) and (iii)

(e) None of the above answers is correct

8. A fair coin is tossed $m+n$ times, where m is **strictly greater** than n. Then the probability of getting m consecutive heads is

(a) $\frac{1}{2^m}$

(b) $\frac{1}{2^n}$

(c) $\frac{m+n}{2^m}$

(d) $(n+1)/2^{m+1}$

(e) None of the above ✓

9. Let $c_1, c_2, \ldots, c_n$ and z be complex numbers such that

$$\frac{1}{z-c_1} + \frac{1}{z-c_2} + \ldots + \frac{1}{z-c_n} = 0.$$

Assume that the numbers $c_1, c_2, \ldots, c_n$ are represented in the complex plane by the vertices of a convex n-gon C. Then

(a) z always lies strictly outside C

(b) z always lies inside or on the boundary of C ✓

(c) z may lie inside or outside C and both are possible

(d) if z lies inside C then z must be the centroid, *i.e.*, $z = \frac{1}{n}\sum_{k=1}^{n} c_k$

(e) None of the above

10. Let $X = (X_1, X_2, X_3)$ be a random vector with covariance matrix Σ. Suppose $a^T X$ and $b^T X$ are uncorrelated for any orthogonal vectors a and b. Then

(a) $\text{rank}(\Sigma) = 1$

(b) $\text{rank}(\Sigma) = 2$

(c) Σ is a scalar multiple of the identity matrix ✓

(d) $\text{rank}(\Sigma) = 3$ but Σ is not necessarily diagonal

(e) None of the above is always true

11. Let the point $X = (X_1, X_2, X_3)$ be chosen uniformly at random from the surface of the 3-dimensional unit ball, i.e., from the set $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. Then

(a) $\mathbb{E}(X_1^4) = \frac{1}{2}$

(b) $\mathbb{E}(X_1^4) = \frac{1}{3}$

(c) $\mathbb{E}(X_1^4) = \frac{1}{5}$ ✓

(d) $\mathbb{E}(X_1^4) = \frac{1}{9}$

(e) None of the above

12. Let X, Y, Z be random variables defined on the same probability space, taking values in a finite set S. Assume that $|S| > 10$ and also that for all $s_1, s_2, s_3 \in S$, $\Pr[X = s_1, Y = S_2, Z = s_3]$ is **strictly positive**.

Under the above assumptions, consider the following statements.

(i) If X is independent of Y, and Y is independent of Z, then X is independent of Z.

(ii) If X and Y are independent, then X and Y are independent conditioned on Z.

(iii) If X and Y are independent conditioned on Z, then X and Y are independent.

Given the above setup, choose the correct statement from the following.

(a) Only statement (i) is always true

(b) Only statement (ii) is always true

(c) Only statement (iii) is always true

(d) Statements (i), (ii), and (iii) are all always true

(e) None of the statements (i), (ii)**, and** (iii) **is always true** ✓

13. Let X be a random variable satisfying $0 \leq X \leq 1$, and let $p = \mathbb{E}[X]$.

Consider the following statements:

P1: $\mathbb{E}[X^2] = p^2$ iff X is constant almost surely.

P2: $\mathbb{E}[X^2] = p$ iff X takes only the values 0 and 1 almost surely.

P3: For every such X, one always has $\mathbb{E}[X^2] \in [p^2, p]$.

Which of the following is true?

(a) Only P1 and P2 are true

(b) Only P2 and P3 are true

(c) Only P1 and P3 are true

(d) All of P1, P2, and P3 are true ✓

(e) None of the statements P1, P2, or P3 is true

14. Which of the following is true in 4 dimensions?

(a) The origin centred Euclidean ball of volume 1 is a subset of the origin centred axis-aligned cube of volume 1

(b) The origin centred Euclidean ball of radius 1 **is a subset of a origin centered cube of volume** $16\frac{1}{2}$ ✓

(c) The origin centred Euclidean ball of radius 1 contains an origin centered cube of side length $\frac{5}{4}$

(d) The volume of a Euclidean ball of radius 1 is $\frac{4}{3}\pi$

(e) The volume of a Euclidean ball of radius 2 is $\frac{64}{3}\pi$

15. Let X be a random variable uniformly distributed in a regular hexagon of side length 1. Let Δ be the difference between the highest eigenvalue of the matrix $\mathbb{E}[XX^T]$ and the lowest eigenvalue of the same matrix. Which of the following is **TRUE**?

(a) $\Delta = 0$ ✓

(b) $\Delta = 1$

(c) $\Delta = 2$

(d) $\Delta = 3$

(e) $\Delta = 4$