Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- For $x > 0$, $\log(x)$ denotes the natural logarithm of x.
- For a metric space (X, d), with $x \in X$ and $r > 0$, $B(x, r)$ denotes the open ball $\{y \in X \mid d(x, y) < r\}$.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm of x, i.e., the distance from x to 0.
- All rings are associative and assumed to contain a multiplicative identity. A ring homomorphisms is assumed to map the multiplicative identity to the multiplicative identity.
- For a ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id or by Id_n. For a matrix $A \in \mathrm{M}_n(R)$, $A^\top$ will stand for its transpose, $\mathrm{trace}(A)$ for its trace, and $\det(A)$ for its determinant.
- For a ring R, $\mathrm{GL}_n(R) \subset \mathrm{M}_n(R)$ denotes the multiplicative group of invertible $n \times n$ matrices with entries in R.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space, equipped with the usual Euclidean topology. Subsets of $\mathrm{M}_n(\mathbb{R})$ are given the subspace topology.
- For a ring R, $R[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables $x_1, \ldots, x_n$ over R.
- For a compact metric space X, $C(X, \mathbb{R})$ denotes the real vector space of continuous real valued functions on X. We define a metric d on $C(X, \mathbb{R})$ by $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.
- If A is a set, $\#A$ stands for the cardinality of A. We denote $\#A = \infty$ if A is infinite.
- If B is a subset of a set A, we write $A \backslash B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a group, and let $S \subseteq G$. The subgroup of G generated by S is defined to be the smallest subgroup of G that contains S.

GS 2026, Mathematics: Stage I Questions

Part A - MCQ.

(1) For $\alpha > 0$, and for all positive integers n, let

$$a_n = \log(1 + n^{-\alpha}).$$

Then,

(a) $\sum_{n=1}^{\infty} a_n$ is convergent for all $\alpha > 0$.
✓ (b) $\sum_{n=1}^{\infty} a_n$ is convergent for all $\alpha > 1$.
(c) $\sum_{n=1}^{\infty} a_n$ is convergent for all $0 < \alpha \leqslant 1$.
(d) $\sum_{n=1}^{\infty} a_n$ is divergent for all $\alpha > 1$.

(2) What is the value of

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{k}{k+n}\right)^{\frac{1}{n}} ?$$

(a) $1/e$
(b) $1/2$
(c) $1/\log(4)$
✓ (d) $1/4$

(3) For $x \in [0,1]$, let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Then

(a) f is continuous but nowhere differentiable.
(b) f is differentiable on $(0,1)$ and its derivative extends continuously to $[0,1]$.
✓ (c) f is differentiable on $(0,1)$ but the derivative diverges as $x \to 1^-$.
(d) f can be extended to a real analytic function on $(0, 1+\epsilon)$ for some $\epsilon > 0$.

(4) For each positive integer n and each $x \in [0,1]$, define

$$f_n(x) = \sin(n)\, \exp\big(-n^4(x - \tfrac{1}{n})^2\big).$$

Which of the following is true about the sequence of functions (f_n) on $[0,1]$?

(a) (f_n) converges to 0 uniformly on $[0,1]$.
✓ (b) (f_n) converges to 0 pointwise, but not uniformly, on $[0,1]$.
(c) (f_n) does not converge pointwise on $[0,1]$.
(d) (f_n) converges uniformly to a nonzero function on $[0,1]$.

(5) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Consider the statements:

(i) If $\lim_{x \to \infty} f(x) = \infty$, then f is not uniformly continuous.

(ii) If f is bounded, then f is uniformly continuous.

Which of the following is correct?

(a) Both (i) and (ii) are true.

(b) (i) is true, (ii) is false.

(c) (i) is false, (ii) is true.

✓ (d) Both (i) and (ii) are false.

(6) Consider the set

$$\mathcal{A} = \{A \in \mathrm{M}_{10}(\mathbb{R}) \mid A^{10} = 0\}.$$

For each $A \in \mathrm{M}_{10}(\mathbb{R})$, define the linear transformation $T_A : \mathrm{M}_{10}(\mathbb{R}) \to \mathrm{M}_{10}(\mathbb{R})$ by $T_A(X) = AX$. Then which of the following is true?

✓ (a) For all $A \in \mathcal{A}$, we have $\det(T_A) = 0$ and $\mathrm{trace}(T_A) = 0$.

(b) There exists $A \in \mathcal{A}$ for which $\det(T_A) \neq 0$ whereas $\mathrm{trace}(T_A) = 0$.

(c) There exists $A \in \mathcal{A}$ for which $\det(T_A) = 0$ whereas $\mathrm{trace}(T_A) \neq 0$.

(d) There exists $A \in \mathcal{A}$ for which $\det(T_A) \neq 0$ and $\mathrm{trace}(T_A) \neq 0$.

(7) Let (X, d) be a complete metric space and U be a proper dense subset of X. Let $f : U \to \mathbb{R}$ be a function. Consider the following two statements:

(i) If f is continuous, then it can be extended to a continuous function $g : X \to \mathbb{R}$.

(ii) If f is uniformly continuous, then it can be uniquely extended to a continuous function $g : X \to \mathbb{R}$.

Then which of the following is correct?

(a) Both (i) and (ii) are true.

(b) Both (i) and (ii) are false.

(c) (i) is true but (ii) is false.

✓ (d) (ii) is true but (i) is false.

(8) Let (X, d) be a metric space with the additional property that

$$d(x, y) \leqslant \max\{d(x, z), d(y, z)\} \text{ for all } x, y, z \in X.$$

Consider the following statements:

(i) For all $x, y, z \in X$, at least two of the numbers $d(x, y), d(y, z), d(x, z)$ are equal.

(ii) For each positive real number r, and $x, y \in X$, the sets $B(x, r)$ and $B(y, r)$ are either equal or disjoint.

Which of the following is true?

✓ (a) (i) and (ii) are true.
(b) (i) and (ii) are false.
(c) (i) is true and (ii) is false.
(d) (ii) is true and (i) is false.

(9) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, such that $T \neq 0$, and $\mathbb{R}^n = (\text{range}(T)) \oplus (\text{null space}(T))$. Consider the following statements:
(i) T is diagonalisable.
(ii) There does not exist any integer $k > 0$ such that $T^k = 0$.
Which of the following must be true?
✓ (a) (i) is false and (ii) is true.
(b) (i) is true and (ii) is false.
(c) (i) and (ii) are true.
(d) (i) and (ii) are false.

(10) For any differentiable function $g : \mathbb{R} \to \mathbb{R}$, let

$$A_g = \{x \in [0,1] \mid g(x) = 0\},$$

$$B_g = \{x \in [0,1] \mid g'(x) = 0\}.$$

Let $\mathcal{S} = \{g : \mathbb{R} \to \mathbb{R} \mid g \text{ is differentiable and } A_g \cap B_g = \varnothing\}$. Which of the following statements is true?
(a) For all $g \in \mathcal{S}$, the set B_g is finite.
✓ (b) For all $g \in \mathcal{S}$, the set A_g is finite.
(c) There exists $g \in \mathcal{S}$ such that both A_g and B_g are infinite.
(d) None of the other statements is true.

(11) Let $g : [1, \infty) \to \mathbb{R}$ be a function such that

$$g(1) = 1 \quad \text{and} \quad g'(x) = \frac{1}{x^2 + g^2(x)}, \text{ for all } x \geqslant 1.$$

Then which of the following statements is true?
(a) $\lim_{x\to\infty} g(x)$ exists, and belongs to $(2, \infty)$.
✓ (b) $\lim_{x\to\infty} g(x)$ exists, and belongs to $(-\infty, 2]$.
(c) None of the other statements is true.
(d) $\lim_{x\to\infty} g(x) = \infty$.

(12) Let A, B be linear endomorphisms of a finite dimensional vector space over a field. Consider the following statements:
(i) Every eigenvalue of AB is an eigenvalue of BA.
(ii) Every eigenvector of AB is an eigenvector of BA.
Which of the following is correct?
(a) Both (i) and (ii) are true.
✓ (b) (i) is true, (ii) is false.
(c) (i) is false, (ii) is true.

(d) Both (i) and (ii) are false.

(13) For a real number a, let $\lfloor a \rfloor$ denote the greatest integer that is less than or equal to a, and let $\{a\} = a - \lfloor a \rfloor$. Let $n \geqslant 4$ be an integer. Then the expression

$$\left\lfloor \frac{1}{\{n!e\}} \right\rfloor$$

equals

✓ (a) n.
(b) $n - 1$.
(c) $\lfloor (n-2)e \rfloor$.
(d) None of the other options.

(14) Consider the space

$$X = \left\{ A \in \mathrm{M}_2(\mathbb{R}) \,\middle|\, AA^\top = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \right\}.$$

Then the number of connected components of X equals

(a) 0.
(b) 1.
✓ (c) 2.
(d) 4.

(15) Let V be the real vector space of all polynomials in $\mathbb{R}[x]$ of degree less than or equal to 10. Consider the linear operator $T : V \to V$ defined by

$$(Tp)(x) = p(2x + 1), \text{ for all } p \in V.$$

What is the trace of T?

(a) 11.
✓ (b) 2047.
(c) 121.
(d) None of the other options.

(16) Let $1 \leqslant p < q$ be integers, and let $f(x, y) = y^p - x^q$. Consider the complex vector space

$$V = \frac{\mathbb{C}[x, y]}{(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})}.$$

Then the dimension of V over $\mathbb{C}$ is:

(a) pq.
✓ (b) $pq - p - q + 1$.
(c) $p + q$.
(d) Infinite.

(17) Let $\Gamma \subset \mathbb{R}^2$ be the graph of $y = \sin(x)$ and let ℓ be a line in $\mathbb{R}^2$ such that $\#(\Gamma \cap \ell) = n \in \mathbb{N}$. Which of the following statements is necessarily true?

(a) n is even.
(b) n is odd.
(c) $n \leqslant 10$.
✓ (d) None of the remaining three options.

(18) Suppose that $f : (-1,1) \to \mathbb{R}$ is infinitely differentiable, and

$$|f(x) - 1 - x - x^2| \leqslant |x|^3$$

for all $x \in (-\frac{1}{3}, \frac{1}{3})$. Then,

(a) $f(0) = f'(0) = f''(0) = 1$.
(b) $f(0) = f'(0) = 1$ and $f'''(0) = 6$.
✓ (c) $f(0) = f'(0) = 1$ and $f''(0) = 2$.
(d) None of the other statements is true.

(19) Consider the following two assertions about a group G.

(I) G is abelian.
(II) The map $\varphi : G \to G$ defined by $\varphi(g) = g^2$ is a group homomorphism.

Which of the following is true?

✓ (a) (I) implies (II), and (II) implies (I).
(b) (I) implies (II), but (II) does not imply (I).
(c) (I) does not imply (II), but (II) implies (I).
(d) (I) does not imply (II), and (II) does not imply (I).

(20) Suppose that $f_n : [0,1] \to [0,1]$ is continuous and increasing for each $n \geqslant 1$, and the sequence $(f_n(x))_{n\geqslant 1}$ of real numbers is decreasing for all $x \in [0,1]$. Let $f : [0,1] \to [0,1]$ be the function defined as $f(x) = \lim_{n\to\infty} f_n(x)$. Then

(a) f is continuous.
✓ (b) f is right-continuous but not necessarily left-continuous.
(c) f is left-continuous but not necessarily right-continuous.
(d) None of the other assertions is true.

Part B - T/F.

T (1) If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and $S \subset \mathbb{R}$ is a bounded subset, then there exists a constant $C_S > 0$ such that $|f(x) - f(y)| \leqslant C_S|x - y|$ for all $x, y \in S$.

T (2) There exists a continuous surjection from $\mathbb{Q}$ to $\{0, 1, \ldots, 2026\}$.

F (3) The subset

$$X = \{f \in C([0,1], \mathbb{R}) : |f(x) - f(y)| \leqslant |x - y|^{1/2} \text{ for all } x, y \in [0,1]\}$$

of $C([0,1],\mathbb{R})$ is compact.

T (4) There is no continuous bijection $f : \mathbb{R} \to [0,1]$.

F (5) Let A, B be linear endomorphisms of a finite dimensional vector space over a field. If A and B are diagonalizable, then AB is also diagonalizable.

T (6) If $A \in \mathrm{GL}_5(\mathbb{C})$ and A^2 is diagonalizable, then A is diagonalizable.

F (7) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists a positive real number c such that $f(x+c) = f(x)$ for all $x \in \mathbb{R}$. Then the set of all periodic functions is a linear subspace of the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

T (8) If $p(x) \in \mathbb{C}[x]$ is a polynomial which is not equal to x, then the polynomial $p(x) - x$ divides the polynomial $p(p(x)) - x$.

T (9) Let A be a square matrix with non-negative real entries such that the sum of entries in each row of A is 1. Then each eigenvalue of A has absolute value less than or equal to 1.

F (10) Let X be a metric space having exactly 4 points. Then, for some positive integer n, there exists a distance preserving map $f : X \to \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with its Euclidean metric.

T (11) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For each positive integer n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = f(x^n)$. If the sequence of functions (f_n) converges uniformly on $\mathbb{R}$, then f must be a constant function.

T (12) Let X, Y be finite metric spaces. Then X and Y are homeomorphic if and only if the real vector spaces $C(X,\mathbb{R})$ and $C(Y,\mathbb{R})$ are isomorphic.

F (13) Suppose a finite group G is the union of its Sylow subgroups. Then the order of G is a power of some prime number.

F (14) For a 2×2 real matrix A, if the limit

$$\sin(A) = \lim_{N \to \infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{A^{2n-1}}{(2n-1)!}$$

exists in $\mathrm{M}_2(\mathbb{R})$, then all eigenvalues of the limit matrix $\sin(A)$ have absolute value less than or equal to 1.

F (15) Let A be a finite abelian group of order 2025. Then for every subgroup $H \leqslant A$ we have $A \cong H \times A/H$ as groups.

F (16) Consider the map $L : \mathbb{R}[t] \to \mathbb{R}[t]$ given by $L(f) = 3f + 2f'$, where f' denotes the derivative (with respect to t) of the polynomial f. Then L is injective, but not surjective.

T (17) Let $\sigma_1, \ldots, \sigma_{80}$ be elements of the permutation group S_{236} such that σ_i^3 is the identity permutation for each i. Then there exist $1 \leqslant i < j \leqslant 80$ such that σ_i is conjugate to σ_j.

T (18) For a permutation $\sigma \in S_{10}$, let $\mathrm{Fix}(\sigma) \subseteq \{1, 2, \ldots, 10\}$ denote the set of fixed points of σ. Then the average

$$\frac{1}{10!} \sum_{\sigma \in S_{10}} \#\mathrm{Fix}(\sigma)$$

is equal to 1.

F (19) The set

$$X = \{(b, c) \in \mathbb{R}^2 \mid \text{the polynomial } x^2 + bx + c \text{ has distinct roots in } \mathbb{C}\}$$

is connected.

T (20) The integer $1^{6n} + 2^{6n} + 3^{6n} + \ldots + 2025^{6n}$ is divisible by 7 for all integers $n \geqslant 10$.