STEP I 2008 Solutions

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Contents

Question 1

What does it mean to say that a number x is irrational?

It means that we cannot write $x = m/n$ where m and n are integers with $n \neq 0$.

Marks

B1 cao; condone absence of the $n \neq 0$ condition.

Prove by contradiction statements A and B below, where p and q are real numbers. ***A:*** *If pq is irrational, then at least one of p and q is irrational.* ***B:*** *If $p+q$ is irrational, then at least one of p and q is irrational.*

Commentary: *To prove these, we're going to have to use the definition of* ***irrational*** *which we've been asked to state. And to prove an implication "if X then Y"* ***by contradiction****, we assume that X is true but Y is false, and then argue that this is impossible; thus if X is true, Y must also be true.*

We start with statement A.

We assume that pq is irrational, but that "at least one of p and q is irrational" is false, so neither p nor q is irrational. If a number is not irrational, it is rational, i.e., it can be written as a fraction m/n.

In slightly more formal language, we can now start writing out our answer:

We first prove statement A.

Assume that pq is irrational, but neither p nor q is irrational, so that both p and q are rational. But then pq is the product of two rational numbers, so is rational. This contradicts that assumption that pq is irrational. So statement A is true.

Now for statement B we argue similarly.

Assume that $p+q$ is irrational, but neither p nor q is irrational, so that both p and q are rational. But then $p+q$ is the sum of two rational numbers, so is rational. This contradicts the assumption that $p+q$ is irrational. So statement B is true.

Marks

M1 for correctly stating or implying the contrapositive statement for either statement A or B.

M1 (dep) for arguing that pq or $p+q$ must be rational if p and q are both rational.

M1 (dep) for deducing the contradiction.

A1 for a correct argument to the second statement.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: *If p and q are irrational, then $p+q$ is irrational.*

Commentary: *To disprove a statement by means of a counterexample, we only need to find ONE example which fails the statement.*

How could we do this? Well, we're looking for examples of irrationals which in some sense "cancel each other out". A bit of playing will provide an answer.

One example is $p=\sqrt{2}$, $q=-\sqrt{2}$.

Marks
B1 for any correct counterexample.
[Total: 6 marks for this part of the question]

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, π^2-e^2, $\pi^2+\mathrm{e}^2$ is rational.

Commentary: *We've been given lots of hints from statements A and B, along with a warning from statement C. The question is, how to use these? We'll need to try multiplying them and using statement A or adding them and using statement B. We must **not** simply add them and say: "Since e and π are both irrational, so is $\pi+\mathrm{e}$"; as we have counterexamples to statement C, this assertion may not be true.*

We assume that the five given numbers are, indeed, irrational.

We have $(\pi+\mathrm{e})+(\pi-\mathrm{e})=2\pi$, which is irrational (if p is irrational, then so is $2p$). So by statement B, at least one of $\pi+\mathrm{e}$ and $\pi-\mathrm{e}$ is irrational.

Similarly, $(\pi^2+\mathrm{e}^2)+(\pi^2-\mathrm{e}^2)=2\pi^2$, which is irrational. So by statement B again, at least one of $\pi^2+\mathrm{e}^2$ and $\pi^2-\mathrm{e}^2$ is irrational.

Commentary: *Help — what do we do now? We now know that at most two of the four numbers are rational, but we have to reduce this to one. It isn't at all obvious how to prove this directly, so let's try arguing by contradiction again: what if **two** of the numbers were rational?*

Assume that both $\pi+\mathrm{e}$ and $\pi^2-\mathrm{e}^2$ are rational. Then

$$\pi-\mathrm{e}=\frac{\pi^2-\mathrm{e}^2}{\pi+\mathrm{e}}$$

would also be rational. But we know that at least one of $\pi+\mathrm{e}$ and $\pi-\mathrm{e}$ is irrational, so $\pi+\mathrm{e}$ and $\pi^2-\mathrm{e}^2$ cannot both be rational. Similarly, we can't have both $\pi-\mathrm{e}$ and $\pi^2-\mathrm{e}^2$ rational.

Commentary: *The difference of two squares is often helpful. We still haven't used statement A — perhaps it will come in useful? There are two cases to go, though.*

Thus if two of the four numbers are rational, they must be $\pi^2 + \mathrm{e}^2$ and one of $\pi \pm \mathrm{e}$.

Assume that $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ are rational. Then $(\pi + \mathrm{e})^2 = (\pi^2 + \mathrm{e}^2) + 2\mathrm{e}\pi$ is the square of a rational number, so is rational. But then $2\mathrm{e}\pi = (\pi + \mathrm{e})^2 - (\pi^2 + \mathrm{e}^2)$ would be rational, contradicting the irrationality of $\mathrm{e}\pi$. Thus we cannot have both $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ rational.

Similarly, if $\pi^2 + \mathrm{e}^2$ and $\pi - \mathrm{e}$ are both rational, we would have $2\mathrm{e}\pi = (\pi^2 + \mathrm{e}^2) - (\pi - \mathrm{e})^2$ being rational, again a contradiction.

Thus at most one of these four numbers is rational.

Marks

[There may well be alternative methods of proving this result, in which case this mark scheme will need to be adapted appropriately.]

M1 for adding or multiplying two of the given four numbers to attempt to use statement A or B.

M1 (dep) for correctly using one of A or B to deduce that at least one of the four numbers is irrational.

M1 (in the same way) for using a different pair of the given four numbers to make a similar deduction about this pair.

A1 for deducing that at most two of the four numbers can be rational.

M1 for factorising $\pi^2 - \mathrm{e}^2$ *and considering the possibility that* $\pi^2 - \mathrm{e}^2$ *and one of* $\pi \pm \mathrm{e}$ *are both rational.*

M1 (dep) for deriving a contradiction.

M1 (dep) for deducing similarly that not both of $\pi^2 - \mathrm{e}^2$ *and the other of* $\pi \pm \mathrm{e}$ *are rational.*

M1 for attempting to deduce that $\pi^2 + \mathrm{e}^2$ *and* $\pi \pm \mathrm{e}$ *cannot both be rational by finding an equation relating them.*

M1 (dep) for deducing $2\mathrm{e}\pi = (\pi^2 + \mathrm{e}^2) \pm (\pi \mp \mathrm{e})^2$.

M1 (dep) for completing the argument.

M1 for the corresponding equation for the other case.

M1 (dep) for completing the argument for the other case.

A1 for deducing from all of the above that at most one of the four given numbers is rational.

[Total: 14 marks for this part of the question]

Commentary: *It turns out that* e, π, π^2 *and* e^2 *are known to be irrational, and that at most one of* $\pi + \mathrm{e}$, $\pi - \mathrm{e}$ *and* $\mathrm{e}\pi$ *is rational. It is unknown whether any of* $\mathrm{e}\pi$, $\pi \pm \mathrm{e}$ *or* $\pi^2 \pm \mathrm{e}^2$ *is rational.*

Question 2

The variables t and x are related by $t = x + \sqrt{x^2 + 2bx + c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b},$$

and hence integrate $\dfrac{1}{\sqrt{x^2 + 2bx + c}}$.

Commentary: *We are given t as a function of x, so we will use implicit differentiation.*

We have, differentiating the given expression with respect to x:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= 1 + \frac{2x+2b}{2\sqrt{x^2+2bx+c}} \\ &= 1 + \frac{2(x+b)}{2(t-x)} \\ &= \frac{(t-x)+(x+b)}{t-x} \\ &= \frac{t+b}{t-x}.\end{aligned}$$

The required result follows on taking the reciprocal of both sides:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\mathrm{d}t/\mathrm{d}x} = \frac{t-x}{t+b}.$$

Marks
M1 for attempting to either use implicit differentiation or for differentiating the whole expression wrt t.
M1 for correctly differentiating the expression.
M1 for substituting for the square root.
M1 for correctly reaching the required form for $\mathrm{d}t/\mathrm{d}x$.
M1 for taking reciprocals to reach the specified answer.
[Total: 5 marks for finding derivative]

Commentary: *We're now asked to find a difficult integral using this result. There are only two general methods for calculating difficult integrals algebraically: integration by substitution and integration by parts. We've been told to work out $\mathrm{d}x/\mathrm{d}t$ when $t = \cdots$, so it would make sense to try integrating by substitution, using the given substitution. We may not actually have to rearrange the given formula to write x in terms of t; we'll just start by replacing the square root with $t - x$ and see what happens ....*

To find the integral, we use the given substitution for x, yielding:

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+2bx+c}}\,\mathrm{d}x &= \int \frac{1}{t-x}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{1}{t-x}\,\frac{t-x}{t+b}\,\mathrm{d}t \\
&= \int \frac{1}{t+b}\,\mathrm{d}t \\
&= \ln|t+b| + k \\
&= \ln\bigl|x+b+\sqrt{x^2+2bx+c}\bigr| + k \\
&= \ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr) + k
\end{aligned}$$

with the last line following as $x^2+2bx+c > x^2+2bx+b^2 = (x+b)^2$, so the parenthesised expression is positive.

> ***Commentary:*** *Yay, it worked out really nicely! We must remember to use the absolute value for the integral of $1/(t+b)$, as it could seemingly be negative (although it then turns out that it actually can't, but we need to justify this).*

> ***Marks***
>
> *M1 for attempting to use the given substitution and correctly replacing the square root expression.*
>
> *M1 for correctly performing the substitution.*
>
> *M1 for reaching $\ln|t+b|+k$ (award even if absolute value signs missed off or constant missing)*
>
> *M1 for substituting back to reach an expression in terms of x.*
>
> *M2 for replacing the absolute value signs by parentheses with correct justification; M1 for replacing them with incorrect justification; M1 for leaving them in. (M0 if absolute values were not used in the first place.)*
>
> *A1 cao (requires constant of integration; award even if M0 on previous method mark).*
>
> *[Total: 7 marks for finding the integral]*

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x+b>0$ but that your result does not hold in the case $b^2 = c$ if $x+b<0$.

> ***Commentary:*** *Given that $b^2 = c$, we should use this to eliminate c from the integrand. Note that we have to pay attention to the fact that the square root symbol means the **positive** square root, so that $\sqrt{x^2}$ may or may not equal x; the best we can say is that $\sqrt{x^2} = |x|$.*

With $b^2 = c$, we have the integral

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{(x+b)^2}}\,\mathrm{d}x \\ &= \int \frac{1}{|x+b|}\,\mathrm{d}x\end{aligned}$$

We now consider the two cases discussed in the question. Firstly, if $x+b>0$, then we have

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x \\ &= \int \frac{1}{x+b}\,\mathrm{d}x \\ &= \ln(x+b)+k'\end{aligned}$$

(we don't need absolute value signs as $x+b$ is positive), whereas before we had

$$\begin{aligned}\ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr)+k &= \ln\bigl(x+b+\sqrt{(x+b)^2}\bigr)+k \\ &= \ln\bigl(x+b+|x+b|\bigr)+k \\ &= \ln\bigl(x+b+(x+b)\bigr)+k \\ &= \ln(2(x+b))+k \\ &= \ln(x+b)+\ln 2+k\end{aligned}$$

Thus our earlier formula works in the case that $b^2 = c$ and $x+b>0$, where we take $k' = k+\ln 2$ (which we may do, as they are arbitrary constants).

> ***Commentary:*** *Ouch — that required some fiddling with the arbitrary constants! Remember that the arbitrary constants are just that: arbitrary. That means that if we have worked out the same indefinite integral in two different ways and reached the same answer, except for a different constant, then we do have the same integral once we pick appropriate values for our arbitrary constants. This is a subtle yet important point.*

Next, when $x+b<0$, direct integration yields:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x \\ &= \int -\frac{1}{x+b}\,\mathrm{d}x \\ &= -\ln|x+b|+k' \\ &= -\ln(-(x+b))+k'\end{aligned}$$

as $x+b<0$. But now our earlier formula yields

$$\begin{aligned}\ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr)+k &= \ln\bigl(x+b+\sqrt{(x+b)^2}\bigr)+k \\ &= \ln\bigl(x+b+|x+b|\bigr)+k \\ &= \ln\bigl(x+b-(x+b)\bigr)+k \\ &= \ln 0+k\end{aligned}$$

which is not even defined. So the earlier result fails to give any answer in the case $b^2 = c$ when $x + b < 0$.

Marks
M1 for identifying $\sqrt{(\cdots)^2} = |\cdots|$.
M1 for correctly integrating the expression directly in the case $x + b > 0$ *and either reaching the conclusion without any absolute value signs, or leaving the absolute value signs in if this was done earlier in the question.*
A1 (ft) for correctly evaluating the expression derived earlier in the case $x + b > 0$, *ignoring the constant of integration.*
M1 for reaching two expressions which are clearly identical (from the direct integration and the expression), modulo constants of integration.
M1 for explaining or otherwise handling the constant of integration discrepancy.
M1 A1 cao for correctly evaluating the integral in the case $x + b < 0$. *(M1 for reaching* $-\ln|x + b|(+k')$*; A1 cao)*
M1 for correctly deducing that the expression found earlier yields the undefined $\ln 0$ *and thus deducing that the expression fails in the case* $x + b < 0$.
[Total: 8 marks for this part of the question]

Commentary: *This substitution is known as* ***Euler's first substitution***. *There is a similar substitution for the case where* $b^2 > c$. *See* `http://planetmath.org/encyclopedia/EulersSubstitutionsForIntegration.html` *for more information.*

Question 3

Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad. \qquad (*)$$

Commentary: *A good trick with inequalities is to subtract one side from the other, so that we're trying to show that something is $\geqslant 0$ or $\leqslant 0$. Let's see if that helps here.*

We have

$$\begin{aligned} ab + cd - bc - ad &= (a - c)b + (c - a)d \\ &= (c - a)(d - b) \\ &\geqslant 0 \qquad \text{as } c \geqslant a \text{ and } d \geqslant b, \end{aligned}$$

from which $(*)$ follows immediately.

Marks
M1 for factorising or other appropriate method.
M1 for correct logic (condone division by zero for this mark).
A1 cso for correctly deducing the result (do not award if division by zero or confusion between $>$ and $\geqslant$).
[Total: 3 marks for introductory inequality]

(i) *If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy$.*

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

Commentary: *While there is a trivial way of proving that $x^2 + y^2 \geqslant 2xy$ (by subtracting $2xy$ from both sides and observing that $x^2 + y^2 - 2xy = (x - y)^2 \geqslant 0$), we have specifically been told to use $(*)$. So we had better do so if we want the marks.*

The inequality $(*)$ had four different variables, whereas we now have only two, so we will try letting two of them be x and two of them be y. The same sort of trick should work for the next part of the question as well.

Letting $a = b = y$ and $c = d = x$ in $(*)$, which we can do as $x \geqslant y$, yields $y^2 + x^2 \geqslant yx + yx$, that is

$$x^2 + y^2 \geqslant 2xy. \qquad (1)$$

Next, letting $a = b = z$, $c = x$ and $d = y$ in $(*)$ gives

$$z^2 + xy \geqslant zx + zy \qquad (2)$$

as we wanted.

Now adding the inequalities (1) and (2) gives us

$$x^2 + y^2 + z^2 + xy \geqslant 2xy + yz + zx.$$

Subtracting xy from both sides yields our desired result:

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx. \tag{3}$$

Finally, we have now proved inequality (3) when $x \geqslant y \geqslant z$, but we need to show that it is true whatever the values of x, y and z. But the inequality is *symmetric* in x, y and z, meaning that rearranging (permuting) the variables in any way does not change the statement. For example, if we swap x and z, we get

$$z^2 + y^2 + x^2 \geqslant zy + yx + xz,$$

which is exactly the same inequality.

So we can assume that the the values of x, y and z we are given satisfy $x \geqslant y \geqslant z$, without changing the statement of the inequality, and we know that the inequality holds in this case.

Marks
M1 for an appropriate substitution into $(*)$ to deduce (1).
M1 (dep) for justifying why the substitution is valid ($c \geqslant a$ etc.).
M1 for an appropriate substitution into $(*)$ to deduce (2).
***NB**: To get these method marks, $(*)$ **must** be used.*
*M1 for adding the inequalities to deduce (3). (Some form of deduction from the previous inequalities **must** be used for this mark.)*
B2 for some argument about symmetry or otherwise deducing that (3) holds for all x, y, z. (B1 for showing that (3) holds for at least one other ordering of x, y, z.)
[Total: 6 marks for part (i)]

(ii) *Show similarly that the inequality* $\dfrac{s}{t} + \dfrac{t}{r} + \dfrac{r}{s} \geqslant 3$ *holds for all positive r, s and t.*

Commentary: *The main difficulty in this part of the question is to figure out how we can use $(*)$ to help us (mimicking the proof in the first part of the question): it is not immediately clear what we should set a, b, c and d to be. Let's work a bit at a time, and say we want to have $ab = s/t$ and $cd = t/r$; then we could try setting $a = s$, $b = 1/t$, $c = t$ and $d = 1/r$. We then need to ensure that it actually works the way we want it to. But we will also have to make sure that our choice matches the additional requirement that $c \geqslant a$ and $d \geqslant b$, so this will take some care.*

We begin by assuming that $r \geqslant s \geqslant t > 0$, and set $c = r$, $a = s$, $d = 1/s$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{s}{r} + \frac{r}{s} \geqslant 1 + 1. \tag{4}$$

__Commentary:__ This is the first step of the argument. We now need to get the other two fractions on the left hand side and s/r on the right hand side so that things cancel as they did earlier.

Now set $c = s$, $a = t$, $d = 1/t$ and $b = 1/r$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r} + \frac{s}{t} \geqslant 1 + \frac{s}{r}. \tag{5}$$

Adding the inequalities (4) and (5) gives

$$\frac{s}{r} + \frac{r}{s} + \frac{t}{r} + \frac{s}{t} \geqslant 1 + 1 + 1 + \frac{s}{r},$$

so that

$$\frac{r}{s} + \frac{t}{r} + \frac{s}{t} \geqslant 1 + 1 + 1, \tag{6}$$

as we want.

__Commentary:__ We would like to now say that we can rearrange the variables any way we like, as in part (i), and the result still holds. Unfortunately, this is not actually the case, and the correct argument is a lot more subtle.

Now this inequality is true for $r \geqslant s \geqslant t$, and by exactly the same argument it will also hold if $s \geqslant t \geqslant r$ or $t \geqslant r \geqslant s$, by cycling the variables.

If $r \geqslant t \geqslant s$, then we have to start again, but the argument is almost identical.

We set $c = r$, $a = t$, $d = 1/t$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{t}{r} + \frac{r}{t} \geqslant 1 + 1. \tag{7}$$

We now set $c = r$, $a = s$, $d = 1/s$ and $b = 1/t$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r} + \frac{s}{t} \geqslant 1 + \frac{s}{r}. \tag{8}$$

Once again, adding these two inequalities gives inequality (6), and by cycling the variables, it is also true if $t \geqslant s \geqslant r$ or $s \geqslant r \geqslant t$.

We have now shown (6) to be true for all six possible orderings of r, s and t, so it is true for all possible (positive) values of r, s and t.

Marks

M1 for explicitly assuming specific ordering on r, s, t.

M1 for using $(*)$ with appropriate choices of $a, \ldots, d$

M1 (dep) for justifying the validity of these choices.

A1 for deducing an inequality of the form of (4).

M1 for substituting appropriately to derive an equation in the form of (5) — condone failure of $c \geqslant a$ or $d \geqslant b$ for this mark.

A1 for deducing correct inequality (require $c \geqslant a$ and $d \geqslant b$ for this mark).

A1 cso for deducing inequality of the form of (6).

B1 for arguing that the result holds for all cyclic permutations of r, s *and* t. *(Condone arguing that the result holds for all positive* r, s, t *because of symmetry.)*

M1 for repeating the argument to handle the case $r \geqslant t \geqslant s$ *or equivalent.*

M1 for deducing equations of the form (7) *and* (8) *or equivalent.*

A1 cso for completing the argument by cyclic permutation of the variables.

NB: The final four marks can be gained by any argument showing that the inequality (6) *holds for all* r, s, t, *dependent upon having obtained the inequality for some specified ordering using the specified method.*

[Total: 11 marks for part (ii)]

Question 4

A function $\mathrm{f}(x)$ *is said to be* convex *in the interval* $a < x < b$ *if* $\mathrm{f}''(x) \geqslant 0$ *for all* x *in this interval.*

(i) *Sketch on the same axes the graphs of* $y = \frac{2}{3}\cos^2 x$ *and* $y = \sin x$ *in the interval* $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ *is defined for* $0 < x < 2\pi$ *by*

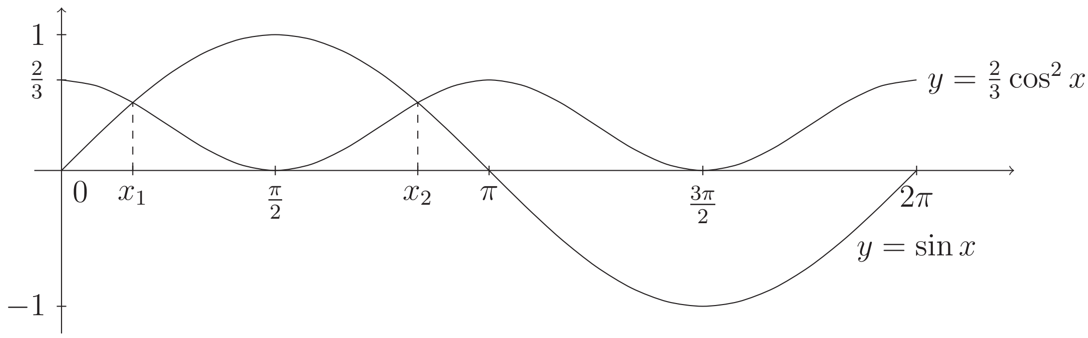
$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ *is convex.*

Commentary: *The sketch is fairly straightforward, remembering that we must indicate any significant points on each of the two graphs.*

To find the intervals where $\mathrm{f}(x)$ *is convex, we must find* $\mathrm{f}''(x)$ *and solve* $\mathrm{f}''(x) > 0$*. This is standard trig inequality work.*

We note that $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ from the double angle formula, so that the graph of $\frac{2}{3}\cos^2 x$ is a translated, stretched version of $y = \cos 2x$.



Now we have

$$\begin{aligned}
\mathrm{f}(x) &= \mathrm{e}^{\frac{2}{3}\sin x} \\
\mathrm{f}'(x) &= (\tfrac{2}{3}\cos x)\mathrm{e}^{\frac{2}{3}\sin x} \\
\mathrm{f}''(x) &= (\tfrac{4}{9}\cos^2 x - \tfrac{2}{3}\sin x)\mathrm{e}^{\frac{2}{3}\sin x} \\
&= \tfrac{2}{3}(\tfrac{2}{3}\cos^2 x - \sin x)\mathrm{e}^{\frac{2}{3}\sin x}.
\end{aligned}$$

As $\mathrm{e}^{\frac{2}{3}\sin x} > 0$ for all x, we have

$$\mathrm{f}''(x) \geqslant 0 \qquad \text{if and only if} \qquad \tfrac{2}{3}\cos^2 x - \sin x \geqslant 0.$$

But we have just drawn a graph of the two functions $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$, so we see that $\mathrm{f}(x) \geqslant 0$ when $0 \leqslant x \leqslant x_1$ and when $x_2 \leqslant x \leqslant 2\pi$, where x_1 and x_2 are the x-coordinates of the points of intersection of the two graphs. So all we need to do is to solve the equation

$$\tfrac{2}{3}\cos^2 x - \sin x = 0$$

to determine the values of x_1 and x_2.

Using $\cos^2 x = 1 - \sin^2 x$ and then factorising gives

$$\begin{aligned}
& \tfrac{2}{3}\cos^2 x = \sin x \\
\iff\quad & \tfrac{2}{3}(1-\sin^2 x) = \sin x \\
\iff\quad & 2 - 2\sin^2 x = 3\sin x \\
\iff\quad & 2\sin^2 x + 3\sin x - 2 = 0 \\
\iff\quad & (2\sin x - 1)(\sin x + 2) = 0 \\
\iff\quad & \sin x = \tfrac{1}{2}
\end{aligned}$$

Therefore $x_1 = \frac{\pi}{6}$ and $x_2 = \frac{5\pi}{6}$, and the function $\mathrm{f}(x)$ is convex in the intervals $0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < 2\pi$.

Marks

B1 for reasonable sketch of sine graph including indications of the maximum and minimum values and at least two of the x-coordinates 0, π and 2π oe.

B2 for $\cos^2$ graph being cos-like: always positive, correct zeros, correct maximum value and reasonably smooth minima at $y = 0$ (no cusps). (B1 if at least three out of four of these.)

M1 for attempting to find $\mathrm{f}''(x)$ to determine intervals in which convex.

A1 for correctly finding $\mathrm{f}'(x)$.

A1 for correctly finding $\mathrm{f}''(x)$. (Can follow through incorrect $\mathrm{f}'(x)$ if the latter has the form (trig fn)$\mathrm{e}^{\frac{2}{3}\sin x}$.)

M1 for correctly identifying required intervals as $0 < x < x_1$ and $x_2 < x < 2\pi$ (condone use of $\leqslant$ rather than $<$).

M1 for reasonable attempt to solve resulting trig equation.

A1 for reaching $\sin x = \frac{1}{2}$.

A1 for correctly finding required intervals (condone use of $\leqslant$ rather than $<$).

[Total: 10 marks for part (i).]

(ii) *The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by*

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \pi/4$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

Commentary: *This looks very similar to part (i). Presumably, therefore, we just do the same: find $\mathrm{g}''(x)$ and determine where it is positive. This should not be too difficult, we hope ...*

We have

$$\begin{aligned}
\mathrm{g}(x) &= \mathrm{e}^{-k\tan x}\\
\mathrm{g}'(x) &= -k\sec^2 x.\mathrm{e}^{-k\tan x}\\
\mathrm{g}''(x) &= (k^2\sec^4 x - 2k\sec^2 x\tan x)\mathrm{e}^{-k\tan x}\\
&= k\sec^2 x(k\sec^2 x - 2\tan x)\mathrm{e}^{-k\tan x}.
\end{aligned}$$

Therefore $g''(x) \geqslant 0$ when $k\sec^2 x - 2\tan x \geqslant 0$, that is, when $k\tan^2 x - 2\tan x + k \geqslant 0$. We can solve the equality $k\tan^2 x - 2\tan x + k = 0$ using the quadratic formula:

$$\begin{aligned}
\tan x &= \frac{2 \pm \sqrt{4-4k^2}}{2k}\\
&= \frac{1 \pm \sqrt{1-k^2}}{k}\\
&= \frac{1 \pm \sqrt{1-\sin^2 2\alpha}}{\sin 2\alpha} && \text{substituting } k = \sin 2\alpha\\
&= \frac{1 \pm \cos 2\alpha}{\sin 2\alpha}.
\end{aligned}$$

Thus we have two possibilities:

$$\tan x = \frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} = \cot\alpha = \tan(\tfrac{\pi}{2}-\alpha)$$

or

$$\tan x = \frac{1-\cos 2\alpha}{\sin 2\alpha} = \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha} = \tan\alpha.$$

Then, since $0 < \alpha < \frac{\pi}{4}$ and $0 < x < \frac{\pi}{2}$, we have $x = \alpha$ or $x = \frac{\pi}{2} - \alpha$.

It follows, since $k > 0$ and $\alpha < \frac{\pi}{2} - \alpha$, that $\mathrm{g}(x)$ is convex, that is, $\mathrm{g}''(x) > 0$, when $0 < x < \alpha$ or when $\frac{\pi}{2} - \alpha < x < \frac{\pi}{2}$. (In more detail, when $x \approx 0$, $\mathrm{g}''(x) \approx k > 0$, and when $x \approx \frac{\pi}{2}$, $\mathrm{g}''(x) \approx k\tan^2 x > 0$.)

Marks

M1 for attempting to find $g''(x)$.

A1 for correctly finding $\mathrm{g}'(x)$.

A1 for correctly finding $\mathrm{g}''(x)$. (Can follow through incorrect $\mathrm{g}'(x)$ if the latter has the form (trig fn)$\mathrm{e}^{-k\tan x}$.)

M1 for deriving quadratic inequality in $\tan x$ which gives $g''(x) \geqslant 0$.

M1 A1 for solving the resulting quadratic equality to find $\tan x$ in terms of 2α.

M2 for using double angle formulæ to simplify this to find expressions for the two possible values of $\tan x$ in terms of α (one mark for each).

A1 cao for determining the two solutions of $\mathrm{g}''(x) = 0$.

A1 cao for determining the intervals in which g is convex (condone use of $\leqslant$ rather than $<$).

SC: If only the interval $0 < x < \alpha$ is correctly found, award A1 A0 for the final two marks.

[Total: 10 marks for part (ii).]

Question 5

The polynomial $\mathrm{p}(x)$ *is given by*

$$x^n + \sum_{r=0}^{n-1} a_r x^r ,$$

where a_0, a_1, $\ldots$, a_{n-1} *are fixed real numbers and* $n \geqslant 1$. *Let* M *be the greatest value of* $|\mathrm{p}(x)|$ *for* $|x| \leqslant 1$. *Then* Chebyshev's theorem *states that* $M \geqslant 2^{1-n}$.

(i) *Prove Chebyshev's theorem in the case* $n = 1$ *and verify that Chebyshev's theorem holds in the following cases:*

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$;

(b) $\mathrm{p}(x) = x^3 - x$.

> ***Commentary:*** *Thankfully, we're not being asked to prove Chebyshev's theorem in general (it's quite hard). But as we are asked to* ***prove*** *it in the case* $n = 1$*, we should write out the statement in this case explicitly; this should make it clearer to us what we are trying to achieve. For the two cases we are asked to* ***verify****, we are not trying to prove a general result of any form, but rather to show that it works in these cases. So we will have to find the greatest absolute value of the given polynomials in the range* $|x| \leqslant 1$.

In the case $n = 1$, Chebyshev's theorem states:

> Let $\mathrm{p}(x)$ be the polynomial $x + a_0$, and let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$. Then $M \geqslant 1$.

If $a_0 > 0$, then when $x = 1$, $p(1) = 1 + a_0 > 1$, so $M > 1$.

If $a_0 < 0$, then when $x = -1$, $p(-1) = -1 + a_0 < -1$, so $|p(-1)| > 1$ and $M > 1$.

Finally, if $a_0 = 0$, then $p(x) = x$, so $|p(x)| = |x|$. It follows that $|p(x)| = |x| \leqslant 1$ when $|x| \leqslant 1$ and $p(1) = 1$, so $M = 1$.

Thus in all cases $M \geqslant 1$.

Now to verify the theorem in the specified cases. The obvious approach is to find the maximum absolute value of the function over the interval. It is important to verify that the function's maximum absolute value really is at least 2^{1-n} in both cases.

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$ is a quadratic whose minimum value is at $x = 0$. So we only need to consider the value of $\mathrm{p}(0)$ and the values of $\mathrm{p}(x)$ at the endpoints of the interval: $\mathrm{p}(-1)$ and $\mathrm{p}(1)$. We have $\mathrm{p}(0) = -\frac{1}{2}$, $\mathrm{p}(-1) = \mathrm{p}(1) = \frac{1}{2}$, so $-\frac{1}{2} \leqslant \mathrm{p}(x) \leqslant \frac{1}{2}$, and hence $|\mathrm{p}(x)| \leqslant \frac{1}{2} = 2^{-1}$ with the maximum value taken on by $|p(x)|$ being $\frac{1}{2}$.

In this case, where $n = 2$, Chebyshev's theorem states that $M \geqslant 2^{-1}$. Since we have $M = 2^{-1}$ in this case, Chebyshev's theorem holds.

(b) Given $\mathrm{p}(x) = x^3 - x$ we first look for stationary points. We have $\mathrm{p}'(x) = 3x^2 - 1$ so there are stationary points at $x = \pm 1/\sqrt{3}$. We thus evaluate $\mathrm{p}(x)$ at these points and at the endpoints $x = \pm 1$. We have $\mathrm{p}(-1) = \mathrm{p}(1) = 0$, $\mathrm{p}(-1/\sqrt{3}) = 2/3\sqrt{3}$ and $\mathrm{p}(1/\sqrt{3}) = -2/3\sqrt{3}$. Hence $|\mathrm{p}(x)| \leqslant 2/3\sqrt{3}$, so that $M = 2/3\sqrt{3}$.

As $n = 3$, we wish to show that $M \geqslant \frac{1}{4}$. But $M^2 = \frac{4}{27} > \frac{4}{64} = \frac{1}{16}$, so $M > \frac{1}{4}$ as required.

A second approach, which is simpler and more direct, is to observe that all we need to do is to find *some* value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x)| \geqslant 2^{1-n}$, for then we know that the *maximum* value of $|p(x)|$ in this interval will be at least that. In (a), we have $|p(1)| = \frac{1}{2} \geqslant 2^{-1}$ and in (b), $|p(\frac{1}{2})| = |\frac{1}{8} - \frac{1}{2}| = \frac{3}{8} \geqslant 2^{-2}$. So we are done.

Marks

M1 for stating Chebyshev's theorem in the case $n = 1$ (may be implied).

M1 for splitting into sensible cases according to the value of a_0.

A1 for showing that the theorem holds in at least one case.

A1 cso for showing that it holds in all cases.

First approach:

Part (a):

M1 for considering local extrema of $p(x)$.

M1 for considering endpoints.

A1 cso for showing that Chebyshev holds in this case.

Part (b):

M1 for finding local extrema of $p(x)$.

M1 for considering endpoints.

A1 for finding maximum value of $|p(x)|$.

A1 cso for showing that Chebyshev holds in this case.

Second approach:

M1 for stating or implying that it is sufficient to find a single point for which $|p(x)| \geqslant 2^{1-n}$.

M1 for justifying this assertion.

M1 for considering $p(1)$ or similar for part (a).

A1 cso for showing that Chebyshev holds in part (a).

M2 for considering $p(\frac{1}{2})$ or similar for part (b). (M1 dependent upon first method mark for this approach for considering the value of $|p(x)|$ for any x in the interval.)

A1 cso for showing that Chebyshev holds in part (b).

[Total: 11 marks for part (i)]

(ii) *Use Chebyshev's theorem to show that the curve $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.*

Commentary: *This would come into the realm of Chebyshev's theorem if it began*

$y = x^5 + \cdots$. Since dividing by a positive constant does not affect the presence of turning points or their nature, we can let $\mathrm{p}(x)$ be the more convenient $y/64$.

Let $\mathrm{p}(x) = x^5 + \frac{1}{64}(25x^4 - 66x^3 - 24x^2 + 3x + 1) = y/64$. The turning points of $\mathrm{p}(x)$ are the same as the turning points of y. Then if we let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$, we have $M \geqslant 2^{-4} = \frac{1}{16}$. Let x_0 be the value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x_0)| = M$, i.e., where $|p(x)|$ takes its maximum value. (If there is more than one such point, choose any of them to be x_0.)

Now $p(-1) = \frac{1}{64}$ and $p(1) = \frac{3}{64}$, and so $|p(1)| < \frac{1}{16}$ and $|p(-1)| < \frac{1}{16}$. But since $|p(x_0)| \geqslant \frac{1}{16}$, we cannot have $x_0 = \pm 1$, so $-1 < x_0 < 1$.

Then x_0 must be a local maximum or a local minimum: if $p(x_0) \geqslant \frac{1}{16}$ then it is at the the greatest value of $p(x)$ in the interval $-1 \leqslant x \leqslant 1$ (and is not at an endpoint); similarly, if $p(x_0) \leqslant -\frac{1}{16}$, then x_0 is at the least value. Either way, x_0 is a turning point as we wanted.

Marks

M1 for considering $p(x) = y/64$.

M1 for stating that the turning points of $p(x)$ are the same as for y.

M1 for evaluating $p(\pm 1)$.

M1 (dep) for stating that $|p(\pm 1)| < \frac{1}{16}$.

M1 for deducing that there is some $-1 < x < 1$ (strict inequality) with $|p(x)| \geqslant \frac{1}{16}$ or that the maximum value of $|p(x)|$ in the closed interval $-1 \leqslant x \leqslant 1$ is obtained in this open interval.

M2 for arguing that either the maximum or minimum value of $p(x)$ in the interval $-1 \leqslant x \leqslant 1$ occurs at this interior point. (M1 if only show one of the two cases.)

A2 cso for deducing that there must be turning point in this interval (A1 if claim that the x_0 found earlier is such a turning point without justification or if only awarded M1 for previous mark).

[Total: 9 marks for part (ii)]

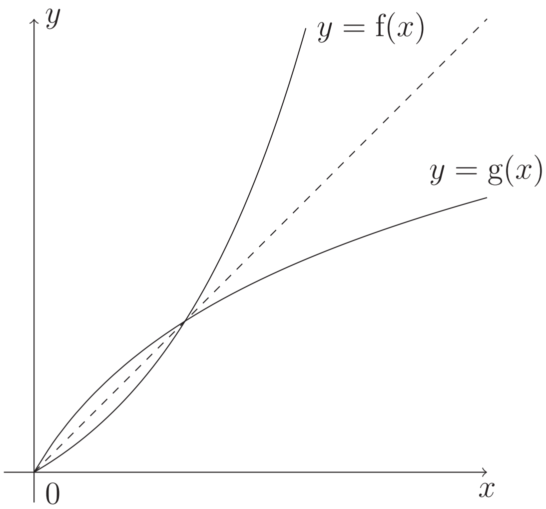
Question 6

The function f *is defined by*

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g *is the inverse function to* f, *so that* $\mathrm{g}(\mathrm{f}(x)) = x$. *Sketch* $\mathrm{f}(x)$ *and* $\mathrm{g}(x)$ *on the same axes.*

Commentary: *This looks like a nasty function at first glance, but on a slightly closer examination, it's not as bad as all that: the* $\mathrm{e} - 1$ *in the denominator is simply a constant, so the function* $\mathrm{f}(x)$ *is just* $\mathrm{e}^x - 1$ *scaled in the* y*-direction. For the inverse function* $\mathrm{g}(x)$, *we reflect the graph of* $y = \mathrm{f}(x)$ *in the line* $y = x$.



Marks

B1 for $\mathrm{f}(x)$ *a convex function.*
B1 for $\mathrm{f}(0) = 0$.
B1 (dep) for $\mathrm{f}'(0) < 1$, *i.e., the graphs of* $\mathrm{f}(x)$ *and* $\mathrm{g}(x)$ *intersect for some* $x > 0$.
B1 (ft) for $y = \mathrm{g}(x)$ *being a reflection of* $y = \mathrm{f}(x)$ *in* $y = x$.
[Total: 4 marks for this part]

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^k \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}} + 1)},$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}} + 1}$, *and explain this result by means of a diagram.*

Commentary: *We'll have to work out the function* $\mathrm{g}(x)$ *in order to be able to do the second integral.*

We find $\mathrm{g}(x)$, the inverse of $\mathrm{f}(x)$, as follows, noting that $y = \mathrm{f}(x)$ if and only if $x = \mathrm{g}(y)$:

$$\begin{aligned} & y = \mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \\ \iff \quad & (\mathrm{e}-1)y = \mathrm{e}^x - 1 \\ \iff \quad & \mathrm{e}^x = (\mathrm{e}-1)y + 1 \\ \iff \quad & x = \ln\big((\mathrm{e}-1)y + 1\big) \end{aligned}$$

so that $\mathrm{g}(x) = \ln\big((\mathrm{e}-1)x + 1\big)$.

Now we can evaluate the integrals. We have

$$\begin{aligned} \int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}\,\mathrm{d}x \\ &= \frac{1}{\mathrm{e}-1}\Big[\mathrm{e}^x - x\Big]_0^{\frac{1}{2}} \\ &= \frac{1}{\mathrm{e}-1}\big((\mathrm{e}^{\frac{1}{2}} - \tfrac{1}{2}) - (1 - 0)\big) \\ &= \frac{1}{\mathrm{e}-1}(\mathrm{e}^{\frac{1}{2}} - \tfrac{3}{2}) \\ &= \frac{2\sqrt{\mathrm{e}} - 3}{2(\mathrm{e}-1)}. \end{aligned}$$

For $\mathrm{g}(x)$, we can either use substitution and the standard result $\int \ln x\,\mathrm{d}x = x\ln x - x + c$ or integration by parts, effectively deriving the result. We demonstrate both methods.

Using substitution, we set $u = (\mathrm{e}-1)x+1$. When $x = 0$, $u = 1$, and when $x = k = 1/(\sqrt{\mathrm{e}}+1)$, we can easily calculate that $u = (\sqrt{\mathrm{e}} - 1) + 1 = \sqrt{\mathrm{e}}$. Finally, $\mathrm{d}u/\mathrm{d}x = \mathrm{e} - 1$. Thus

$$\begin{aligned} \int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \int_0^k \ln\big((\mathrm{e}-1)x + 1\big)\,\mathrm{d}x \\ &= \int_1^{\sqrt{\mathrm{e}}} \ln u \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \int_1^{\sqrt{\mathrm{e}}} \frac{1}{\mathrm{e}-1} \ln u\,\mathrm{d}u \\ &= \frac{1}{\mathrm{e}-1}\Big[u\ln u - u\Big]_1^{\sqrt{\mathrm{e}}} \\ &= \frac{1}{\mathrm{e}-1}\big((\sqrt{\mathrm{e}}\ln\sqrt{\mathrm{e}} - \sqrt{\mathrm{e}}) - (\ln 1 - 1)\big) \\ &= \frac{1}{\mathrm{e}-1}(1 - \tfrac{1}{2}\sqrt{\mathrm{e}}) \\ &= \frac{2 - \sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}. \end{aligned}$$

Alternatively we can use integration by parts. We first note that

$$\begin{aligned}
\ln((\mathrm{e}-1)k+1) &= \ln\left(\frac{\mathrm{e}-1}{\sqrt{\mathrm{e}}+1}+1\right) \\
&= \ln\big((\sqrt{\mathrm{e}}-1)+1\big) \\
&= \ln(\sqrt{\mathrm{e}}) \\
&= \tfrac{1}{2}.
\end{aligned}$$

Then we can evaluate our integral as follows:

$$\begin{aligned}
\int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \int_0^k 1.\ln((\mathrm{e}-1)x+1)\,\mathrm{d}x \\
&= \big[x\ln((\mathrm{e}-1)x+1)\big]_0^k - \int_0^k x\left(\frac{\mathrm{e}-1}{(\mathrm{e}-1)x+1}\right)\mathrm{d}x \\
&= k\ln((\mathrm{e}-1)k+1) - \int_0^k \frac{(\mathrm{e}-1)x+1-1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \int_0^k 1 - \frac{1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \left[x - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)x+1)\right]_0^k \\
&= \tfrac{1}{2}k - \left(\left(k - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)k+1)\right) - (0-\ln 1)\right) \\
&= \tfrac{1}{2}k - \left(k - \frac{1}{2(\mathrm{e}-1)}\right) \\
&= \frac{1}{2(\mathrm{e}-1)} - \tfrac{1}{2}k \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{1}{2(\sqrt{\mathrm{e}}+1)} \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}
\end{aligned}$$

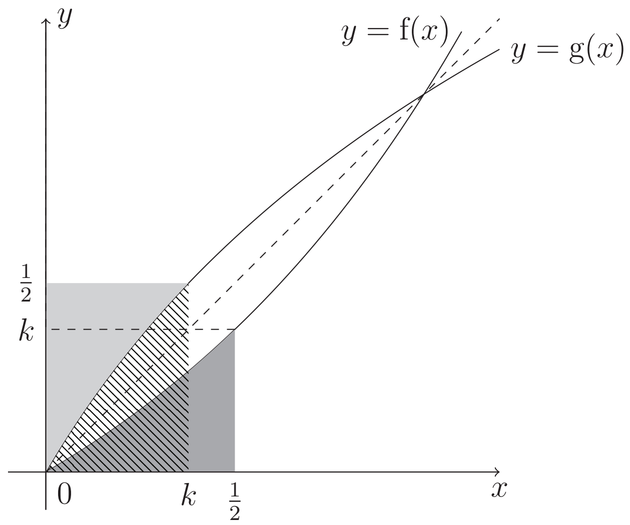
as we found using the substitution method.

We therefore have

$$\begin{aligned}
\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \frac{2\sqrt{\mathrm{e}}-3}{2(\mathrm{e}-1)} + \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)} \\
&= \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{1}{2(\sqrt{\mathrm{e}}+1)}
\end{aligned}$$

as we wanted.

Finally, to explain this result with the aid of a diagram, we want to fill in the two areas indicated by the integrals on our sketch of the functions above. It would be useful to know the value of $f(\frac{1}{2})$ for this purpose: it is $(e^{1/2} - 1)/(e - 1) = 1/(\sqrt{e} + 1) = k$, which is very convenient. Since $g(x)$ is the inverse of $f(x)$, it follows likewise that $g(k) = \frac{1}{2}$. We can now sketch the areas on our graph.



In this sketch, the dark shaded area is the integral $\int_0^{1/2} f(x)\,dx$ and the striped area is the integral $\int_0^k g(x)\,dx$. We have reflected the dark shaded area in the line $y = x$ to get the light shaded area also shown. It is now clear that the shaded and striped areas add to give a $\frac{1}{2} \times k$ rectangle, so the area is $k/2 = 1/2(\sqrt{e} + 1)$, as we found.

Marks

M1 for rearranging $y = f(x)$ to $e^x = \cdots$.

A1 cao for finding $g(x)$ explicitly.

[Total: 2 marks for finding $g(x)$]

M1 for finding $\int f(x)\,dx$ (indefinite integral).

M1 for correct substitution of limits.

A1 cao for evaluating $\int_0^{1/2} f(x)\,dx$.

[Total: 3 marks for finding $\int_0^{1/2} f(x)\,dx$]

Integral $\int g(x)\,dx$ by substitution method:

M1 for attempting an appropriate substitution.

A1 for correctly performing substitution on integrand.

A1 for correctly performing substitution on limits or for correctly substituting back to get integral in terms of x.

M1 A1 for correctly integrating $\ln u$, either via parts or by quoting result.

A1 for correctly evaluating integral, giving simplified answer with no k in it. (Can get this mark if k's left in and they are dealt with in the calculation of $\int f + \int g$.)

Integral $\int g(x)\,dx$ by parts:

M1 for attempting integration by parts by writing $g(x) = 1.\ln(\cdots)$ or something equally useful.

A1 for completely correct application of parts.

M1 for appropriate method of integrating resulting integral.

A1 for completely correct (indefinite) integral.

M1 for reaching a correct expression for $\int_0^k \mathrm{g}(x)\,\mathrm{d}x$ *in terms of* k*.*

A1 for simplified answer with no k *in it. (Can get this mark if* k*'s left in and they are dealt with in the calculation of* $\int \mathrm{f} + \int \mathrm{g}$*.)*

[Total: 6 marks for finding $\int_0^k \mathrm{g}(x)\,\mathrm{d}x$*]*

A1 cso for evaluating $\int \mathrm{f} + \int \mathrm{g}$*, deducing the given result.*

[Total: 1 mark for evaluating $\int \mathrm{f} + \int \mathrm{g}$*]*

Explanation:

M1 for correctly indicating regions representing the two integrals on a sketch.

M1 for identifying $\int \mathrm{f}$ *region with region above* $y = \mathrm{g}(x)$ *or the* $\int \mathrm{g}$ *region with the region above* $y = \mathrm{f}(x)$*.*

M1 (dep) for concluding that $\int \mathrm{f} + \int \mathrm{g}$ *is the area of a* $\frac{1}{2} \times k$ *rectangle.*

A1 cso for concluding that $\int \mathrm{f} + \int \mathrm{g} = 1/(2(\sqrt{\mathrm{e}} + 1))$ *as required.*

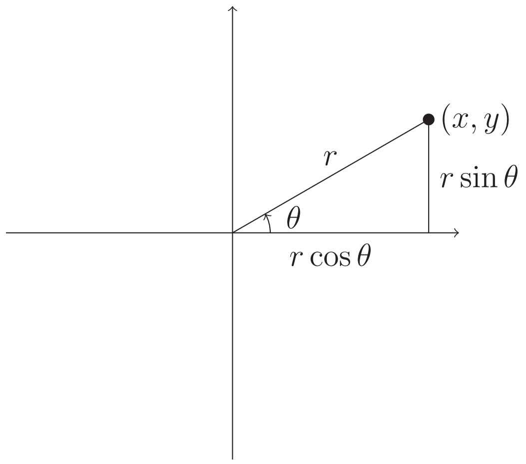
[Total: 4 marks for explanation]

[Total: 16 marks for this part of the question]

Question 7

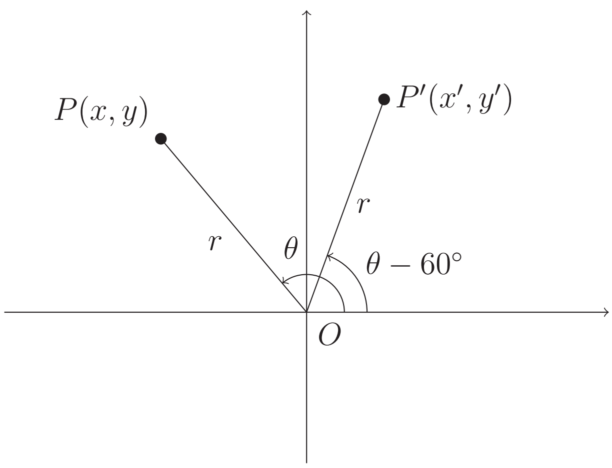
The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by $60°$ clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$. What is the new y-coordinate in the case of an anti-clockwise rotation by $60°$?

Commentary: *The question suggests to us to write $x = r\cos\theta$ and $y = r\sin\theta$, and we should certainly take the examiner's hint! This rewriting corresponds to the following picture, recalling what we know about the* sin *and* cos *functions.*



For the rest of this part, it is definitely worth our while drawing a sketch, so that we can see what is going on.

The situation described is illustrated in the following diagram, where OP' is the image of OP under the specified rotation, P' having coordinates (x', y').



Then we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$y' = r\sin(\theta - 60°)$$

We use the compound angle formula for sine to get

$$\begin{aligned} y' &= r(\sin\theta\cos 60° - \cos\theta\sin 60°) \\ &= \tfrac{1}{2}r\sin\theta - \tfrac{\sqrt{3}}{2}r\cos\theta \\ &= \tfrac{1}{2}y - \tfrac{\sqrt{3}}{2}x \\ &= \tfrac{1}{2}(y - \sqrt{3}\,x). \end{aligned}$$

Likewise, if the rotation is by $60°$ anticlockwise, we replace the $\theta - 60°$ by $\theta + 60°$ and repeat the above to get

$$y' = \tfrac{1}{2}(y + \sqrt{3}\,x).$$

Marks

M1 for an appropriate sketch or other argument which would lead to an expression for y'.

M1 for $y' = r\sin(\theta - 60°)$.

M1 for using an appropriate compound angle formula.

A1 cso for deriving stated expression for y' in terms of x and y.

Alternatively:

M2 for quoting rotation matrix $P = \begin{pmatrix} \cos 60° & \sin 60° \\ -\sin 60° & \cos 60° \end{pmatrix}$.

M1 for $\mathbf{x}' = R\mathbf{x}$.

A1 cso for evaluating the product to reach the stated answer.

Anticlockwise rotation:

M1 for replacing $-60°$ with $+60°$.

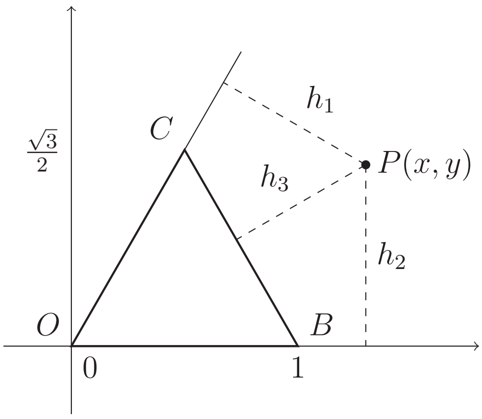
A1 cao for correct answer.

[Total: 6 marks for this part]

An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\left|y - \sqrt{3}\,x\right|$ and find expressions for h_2 and h_3.

Commentary: *We* ***must*** *draw ourselves a picture if we want to stand any chance of understanding what's going on here and how this relates to the first part of the question.*



Clearly $h_2 = |y|$. (We need to take the absolute value as P might lie under the x-axis.)

For h_1, consider rotating the entire shape by 60° clockwise about O. This will rotate OC to the x-axis, and the perpendicular from P to OC will become vertical. The transformed y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$, as we deduced earlier, so $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$.

Finally, for h_3, we start by rotating anticlockwise by 60° around O. Then P ends up with y-coordinate $\frac{1}{2}(y + \sqrt{3}\,x)$ and the side BC ends up lying along the line $y = \frac{\sqrt{3}}{2}$; subtracting these then gives $h_3 = \frac{1}{2}\big|y + \sqrt{3}\,x - \sqrt{3}\big|$.

[An alternative argument is to translate the whole diagram by one unit to the left first, so that B moves to the origin and P moves to $(x - 1, y)$. Then rotating around the new B gives P the new y-coordinate of $\frac{1}{2}(y + \sqrt{3}(x - 1))$, which is the same as before.]

Marks

M1 for rotating clockwise about O to make perpendicular from P to OC vertical. (Allow other correct methods for this mark.)

A1 cso for deducing h_1.

B1 for h_2 (must have an absolute value to gain this mark, not just $h_2 = y$).

M1 for rotating anticlockwise about O and describing the new configuration.

M1 for deducing equation of the rotated BC or equivalent method.

A1 for deducing h_3 (including absolute value).

NB Penalise lack of absolute value signs at most once in this part of the question.

[Total: 6 marks for this part of the question]

Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

We have

$$\begin{aligned} h_1 + h_2 + h_3 &= \tfrac{1}{2}\big(|2y| + \big|y - \sqrt{3}\,x\big| + \big|y + \sqrt{3}\,x - \sqrt{3}\big|\big) \\ &= \tfrac{1}{2}\big(|2y| + \big|\sqrt{3}\,x - y\big| + \big|\sqrt{3} - \sqrt{3}\,x - y\big|\big). \end{aligned}$$

Commentary: *We've used the identity $|a| = |-a|$ for all a. We'd like to somehow get rid of the absolute value symbols, because then everything would cancel out nicely and* *we'd be left with $\frac{1}{2}\sqrt{3}$ as we want. So we will make use of the* ***triangle inequality****: $|x+y| \leqslant |x|+|y|$ with equality if and only if $x \geqslant 0$ and $y \geqslant 0$ or $x \leqslant 0$ and $y \leqslant 0$.*

The equivalent for three variables is $|x+y+z| \leqslant |x|+|y|+|z|$ with equality if and only if all of x, y and z are $\geqslant 0$ or all of them are $\leqslant 0$.

This is straightforward to prove by considering cases depending on the values of x, y and z.

Using the triangle inequality, we then have

$$\begin{aligned} h_1+h_2+h_3 &= \tfrac{1}{2}\big(|2y| + |\sqrt{3}\,x - y| + |\sqrt{3}-\sqrt{3}\,x-y|\big) \\ &\geqslant \tfrac{1}{2}\big|(2y) + (\sqrt{3}\,x-y) + (\sqrt{3}-\sqrt{3}\,x-y)\big| \\ &= \tfrac{1}{2}\sqrt{3}, \end{aligned}$$

with equality if and only if all of the bracketed terms are $\geqslant 0$ or all of the bracketed terms are $\leqslant 0$.

If all of the terms are negative or zero, then $2y \leqslant 0$, so $y \leqslant 0$. And $\sqrt{3}\,x - y \leqslant 0$ implies that $x \leqslant y/\sqrt{3} \leqslant 0$. But then we must have $\sqrt{3}-\sqrt{3}\,x-y > 0$, which is impossible. So we cannot have all three terms negative or zero.

Therefore, if we have equality, we must have all three terms positive or zero. But $2y \geqslant 0$ if and only if P lies on or above the x-axis, that is, on or above the line OB. Similarly, $\sqrt{3}\,x - y \geqslant 0$ if and only if $y \leqslant \sqrt{3}\,x$, which is true if and only if P lies on or below the line OC (which has equation $y = \sqrt{3}\,x$). Finally, $\sqrt{3}-\sqrt{3}\,x-y \geqslant 0$ if and only if $y \leqslant \sqrt{3}-\sqrt{3}\,x$, which is true if and only if P lies on or below the line BC (which has equation $y = \sqrt{3}-\sqrt{3}\,x$).

Putting these together shows that $h_1+h_2+h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or inside the triangle OBC.

Marks

M1 for rewriting $|y-\sqrt{3}\,x|$ as $|\sqrt{3}\,x-y|$ or $|2y|$ as $|-2y|$.

M1 for attempting to apply the triangle inequality or some similar type of argument.

M2 for deducing algebraic conditions under which equality holds (M1 if only get one of $\geqslant 0$ or $\leqslant 0$, or if only have one direction of implication).

M1 for deducing that we cannot have all terms $\leqslant 0$.

M1 for identifying at least two of the regions $2y \geqslant 0$, $\sqrt{3}\,x - y \geqslant 0$ and $\sqrt{3}-\sqrt{3}\,x-y \geqslant 0$ (condone use of $>$ instead of $\geqslant$ for this mark).

M1 for correctly identifying region in which all three hold.

A1 cso for deducing stated necessary and sufficient conditions for equality to hold.

SC: If only correctly argue P in triangle $\Longrightarrow h_1+h_2+h_3 = \frac{1}{2}\sqrt{3}$:

M1 for identifying at least two of the regions $2y \geqslant 0$, $\sqrt{3}\,x - y \geqslant 0$ and $\sqrt{3}-\sqrt{3}\,x-y \geqslant 0$ algebraically (condone use of $>$ instead of $\geqslant$ for this mark).

A1 for correctly identifying region of triangle (must use $\geqslant$ for this mark).

M1 for deducing values of h_1, h_2, h_3 without absolute value signs in this case.
A1 for deducing value of $h_1 + h_2 + h_3$ in this case.
Then award M0 M0 M0 M0 for remaining four marks.
[Total: 8 marks for this part of the question]

Question 8

> ***(i)*** *The gradient y' of a curve at a point (x, y) satisfies*
>
> $$(y')^2 - xy' + y = 0. \qquad (*)$$
>
> *By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.*
>
> *Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.*

Commentary: *Differentiating $(*)$ is going to require some serious use of the chain and product rules. Recall that $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$, so that $\frac{\mathrm{d}}{\mathrm{d}x}(y') = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y''$.*

Differentiating $(*)$ with respect to x, using the chain rule and product rule, gives

$$2y'y'' - y' - xy'' + y' = 0,$$

which, on cancelling terms and factorising, yields

$$(2y' - x)y'' = 0,$$

so either $y'' = 0$ or $2y' - x = 0$.

We now solve these two differential equations. Firstly, by integrating $y'' = 0$ twice with respect to x, we get $y' = a$, so $y = ax + b$ (where a and b are constants). Substituting this back into $(*)$ gives

$$a^2 - x.a + (ax + b) = 0,$$

so

$$a^2 + b = 0,$$

which gives us the straight line $y = mx + c$ with $m = a$ and $c = b = -a^2 = -m^2$.

In the other case, $y' = \frac{1}{2}x$, which on integrating gives $y = \frac{1}{4}x^2 + c$. Substituting this back into $(*)$ gives

$$(\tfrac{1}{2}x)^2 - x(\tfrac{1}{2}x) + (\tfrac{1}{4}x^2 + c) = 0,$$

so $c = 0$ and $y = \frac{1}{4}x^2$, or $4y = x^2$ as required.

Marks

M1 for differentiating $(y')^2$ using chain rule.

M1 for differentiating xy' using product rule.

A1 cao for correctly differentiating $(*)$.

M1 for factorising and deducing $y'' = 0$ or $2y' - x = 0$.

M1 for integrating $y'' = 0$ twice.

M1 for substituting $y = ax + b$ into $(*)$.

A1 cso for deducing $c = -m^2$.

SC: Integrating $y'' = 0$ only once and substituting $y' = k$ back into $(*)$ to deduce $y = kx - k^2$ without justification that this y satisfies $y' = k$ scores M0 M1 A1 for these last three marks.

M1 for integrating $2y' - x = 0$.

A1 cso for substituting into $(*)$ to deduce $c = 0$.

SC: Substituting $y' = \frac{1}{2}x$ into $(*)$ to find y without justification that this y satisfies $y' = \frac{1}{2}x$ scores M1 A0 for these last two marks.

NB: Penalise substituting y' into $(*)$ without justification at most once in part (i) and at most once in part (ii).

[Total: 9 marks for part (i)]

(ii) *The gradient y' of a curve at a point (x, y) satisfies*

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0. \qquad (\dagger)$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Commentary: *This equation is similar to $(*)$, in that it involves a $(y')^2$ term. So it would make sense to try differentiating it with respect to x, as we did in part (i).*

We also note that just as the derivative of uv is $u'v + uv'$, the derivative of uvw is $u'vw + uv'w + uvw'$.

Differentiating $(\dagger)$ with respect to x, using the chain rule and product rule, gives:

$$2x(y')^2 + (x^2 - 1).2y'y'' - 2yy' - 2xy'y' - 2xyy'' + 2yy' = 0,$$

which, on cancelling terms, yields

$$2(x^2 - 1)y'y'' - 2xyy'' = 0.$$

Finally, factorising brings us to our desired conclusion:

$$\big((x^2 - 1)y' - xy\big)y'' = 0,$$

so either $y'' = 0$ or $y'(x^2 - 1) - xy = 0$.

We now solve these two equations. Again, integrating $y'' = 0$ twice gives $y' = a$, so $y = ax+b$. Substituting this back into $(\dagger)$ gives:

$$(x^2 - 1).a^2 - 2x(ax + b).a + (ax + b)^2 - 1 = 0,$$

which is equivalent to

$$a^2x^2 - a^2 - 2a^2x^2 - 2abx + a^2x^2 + 2abx + b^2 - 1 = 0,$$

so

$$-a^2 + b^2 - 1 = 0,$$

or $b^2 = a^2 + 1$. Thus the equation is satisfied by straight lines $y = mx + c$ where $c^2 = m^2 + 1$.

In the other case, $y'(x^2 - 1) - xy = 0$.

It looks as though we could solve this by separating the variables to give:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{x}{x^2-1}\,\mathrm{d}x,$$

so $\ln y = \frac{1}{2}\ln|x^2-1| + c$. Doubling and exponentiating then gives

$$y^2 = C|x^2-1|.$$

To determine C and if its value depends upon whether $|x|<1$ or $|x|>1$, we ought to try substituting back into (†). However, it is simpler to substitute in the result $y'(x^2-1) = xy$ directly into (†).

Substituting $y' = xy/(x^2-1)$ back into (†) gives

$$(x^2-1)\left(\frac{xy}{x^2-1}\right)^2 - 2xy\left(\frac{xy}{x^2-1}\right) + y^2 - 1 = 0.$$

Expanding brackets then gives

$$\frac{x^2y^2}{x^2-1} - \frac{2x^2y^2}{x^2-1} + y^2 - 1 = 0$$

so

$$-\frac{x^2y^2}{x^2-1} + y^2 - 1 = 0.$$

Multiplying by x^2-1 gives

$$-x^2y^2 + (y^2-1)(x^2-1) = 0$$

so

$$-x^2y^2 + y^2x^2 - y^2 - x^2 + 1 = 0$$

or

$$-x^2 - y^2 + 1 = 0,$$

and we therefore deduce that the only other possible solution is the circle $x^2+y^2=1$.

We must finally check that the circle does, in fact, satisfy $y' = xy/(x^2-1)$. Differentiating $x^2+y^2=1$ with respect to x gives $2x+2yy'=0$, so $y' = -x/y = -xy/y^2 = -xy/(1-x^2)$, as required.

Marks

M1 for attempting to differentiate (†).

M1 for differentiating $(x^2-1)(y')^2$ using product and chain rules.

M1 for differentiating $2xyy'$ using product rule.

A1 for correctly differentiating (†).

M1 for factorising derivative to obtain $y''=0$ or $y'(x^2-1)-xy=0$.

M1 for integrating $y''=0$ and substituting into (†).

A1 for deducing form of straight line. (Condone $c=\sqrt{m^2+1}$ instead of $\pm\sqrt{\cdots}$.)

Separating variables method:

M1 for separating variables and integrating.

M1 for exponentiating to deduce $y^2 = C|x^2 - 1|$ (condone absence of absolute value signs).

M1 for substituting back into (†) to deduce value of C.

A1 for deducing equation of circle.

Substitution method:

M1 for substituting for y' into (†).

M2 for simplifying resulting equation (M1 if attempt simplification but get stuck).

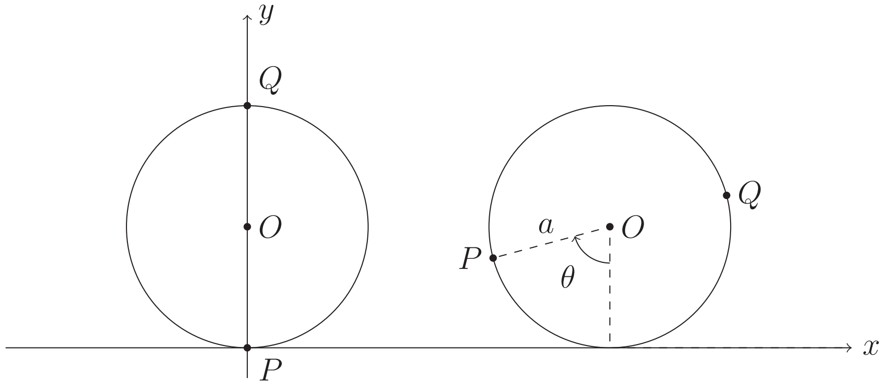
A1 for deducing equation of circle.

Commentary: *What we have in both parts is a family of straight lines satisfying a differential equation, which itself actually determines this family uniquely. The* ***envelope*** *of the straight lines, a curve which is tangent to every one of the straight lines, is then also a solution of the differential equation, and is called a* ***singular solution*** *of the equation.*

Question 9

Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

Commentary: *We **must** draw a diagram to show what is happening if we are to stand any chance of success here.*



The diagram shows the hoop rolling to the right, indicating the positions of the hoop at time 0 and time t.

Taking the origin to be at the initial position of P, the coordinates of the centre of the hoop at time t are $(a\theta, a)$, since the hoop has rolled a distance $a\theta$. Therefore P has coordinates $(a(\theta - \sin\theta), a(1 - \cos\theta))$ and position vector

$$\mathbf{r}_P = a(\theta - \sin\theta)\mathbf{i} + a(1 - \cos\theta)\mathbf{j}.$$

The velocity vector of P is then

$$\begin{aligned}
\dot{\mathbf{r}}_P &= \frac{\mathrm{d}}{\mathrm{d}t}\big(a(\theta - \sin\theta)\big)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}t}\big(a(1 - \cos\theta)\big)\mathbf{j} \\
&= a\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathbf{i} - a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{j} \\
&= a(\dot{\theta} - \cos\theta.\dot{\theta})\mathbf{i} + a\sin\theta.\dot{\theta}\,\mathbf{j} \\
&= a\dot{\theta}\big((1 - \cos\theta)\mathbf{i} + \sin\theta\,\mathbf{j}\big),
\end{aligned}$$

and hence P has speed v_P given by

$$\begin{aligned} v_P^2 &= (a\dot{\theta})^2\big((1-\cos\theta)^2 + (\sin\theta)^2\big) \\ &= (a\dot{\theta})^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) \\ &= (a\dot{\theta})^2(2 - 2\cos\theta) \\ &= (a\dot{\theta})^2.4\sin^2\tfrac{1}{2}\theta \\ &= (2a\dot{\theta}\sin\tfrac{1}{2}\theta)^2, \end{aligned}$$

so that $v_P = 2a|\dot{\theta}\sin\frac{1}{2}\theta|$.

Similarly, the coordinates of Q are $(a(\theta+\sin\theta), a(1+\cos\theta))$, so Q has position vector

$$\mathbf{r}_Q = a(\theta+\sin\theta)\mathbf{i} + a(1+\cos\theta)\mathbf{j}.$$

Arguing as before, the velocity vector of Q is then

$$\begin{aligned} \dot{\mathbf{r}}_Q &= a(\dot{\theta} + \cos\theta.\dot{\theta})\mathbf{i} - a\sin\theta.\dot{\theta}\,\mathbf{j} \\ &= a\dot{\theta}\big((1+\cos\theta)\mathbf{i} - \sin\theta\,\mathbf{j}\big) \end{aligned}$$

so that Q has speed v_Q given by

$$\begin{aligned} v_Q^2 &= (a\dot{\theta})^2\big((1+\cos\theta)^2 + (\sin\theta)^2\big) \\ &= (a\dot{\theta})^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta) \\ &= (a\dot{\theta})^2(2 + 2\cos\theta) \\ &= (a\dot{\theta})^2.4\cos^2\tfrac{1}{2}\theta. \end{aligned}$$

Adding $v_P^2 = (a\dot{\theta})^2.4\sin^2\frac{1}{2}\theta$ to this gives the total kinetic energy as

$$\begin{aligned} \tfrac{1}{2}mv_P^2 + \tfrac{1}{2}mv_Q^2 &= \tfrac{1}{2}m(v_P^2 + v_Q^2) \\ &= \tfrac{1}{2}m\big((a\dot{\theta})^2.4\sin^2\tfrac{1}{2}\theta + (a\dot{\theta})^2.4\cos^2\tfrac{1}{2}\theta\big) \\ &= \tfrac{1}{2}m(a\dot{\theta})^2.4 \\ &= 2ma^2\dot{\theta}^2 \end{aligned}$$

as required.

Marks

M1 for a reasonable sketch of the situation, in particular with the direction of θ compatible with the direction of motion (may be implied).

M1 for finding the position of the centre $(a\theta, a)$.

M1 for finding the x-coordinate of P (ft incorrect centre).

M1 for finding the y-coordinate of P (ft incorrect centre).

A1 cao for coordinates of P.

M1 for calculating the velocity vector or components by differentiation.

A1 (ft) for velocity vector or components.

M1 for finding v_P^2.

M1 for simplifying using double angle formula.

A1 for v_P^2 simplified (may be implied).
A1 for v_P (must have absolute value signs).
M1 for coordinates of Q (allow ft from earlier errors).
A1 cao for correct v_Q^2 (any form).
M1 for attempting to calculate $KE = \frac{1}{2}mv_P^2 + \frac{1}{2}v_Q^2$.
A1 cso for deriving stated result.

Alternative method for the last four marks:
M1 for using $\mathrm{KE}_{total} = \mathrm{KE}_{linear} + \mathrm{KE}_{rotational}$.
M1 for calculating KE_{linear}.
M1 for calculating $\mathrm{KE}_{rotational}$.
A1 cso for deriving the stated result.
[Total: 15 marks for this part.]

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

Commentary: *It is always worthwhile considering energy in mechanics questions. Here it makes our life a breeze.*

Consider the hoop as a single system. The only external forces on the hoop are gravity and the normal reaction. Both of these are vertical, while the hoop only moves in a horizontal direction. Therefore, no work is done on the hoop, so that GPE + KE is constant.

The gravitational potential energy of P and Q together, taking the centre of the hoop as potential energy zero, gives $mga(-\cos\theta) + mga(+\cos\theta) = 0$. So the GPE of the system is constant, meaning that the kinetic energy is also constant.

Since the total kinetic energy is $2ma^2\dot{\theta}^2$, it follows that $\dot{\theta}$ is constant, that is, the hoop rolls with the constant speed $a\dot{\theta}$.

Marks
B1 for stating GPE + KE is constant.
M1 for calculating GPE at angle θ.
M1 (dep) for deducing GPE is constant.
A1 for deducing KE is constant.
A1 for deducing $\dot{\theta}$ is constant, so rolls at constant speed.
[Total: 5 marks for this part]

Question 10

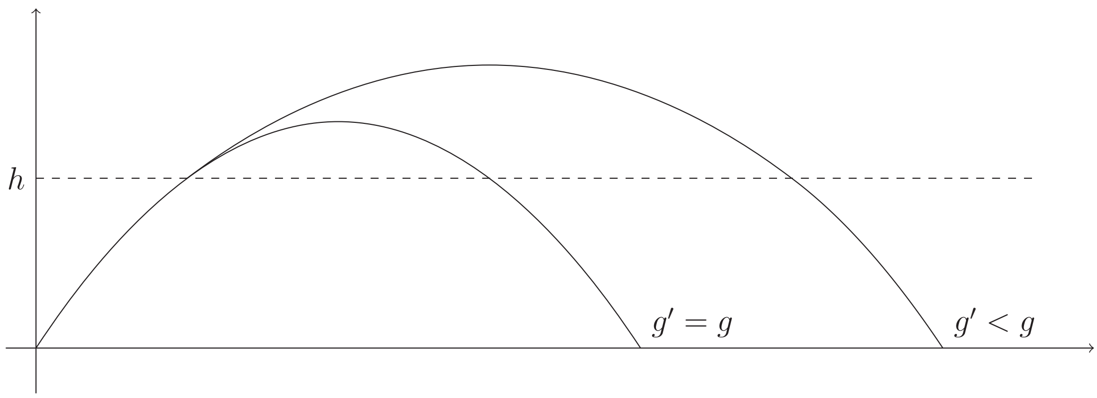
On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2 \sin^2 \alpha > 2gh$. Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

We know that the path of a projectile is parabolic when the gravity is constant. When the gravity is less, the parabola will be "bigger", as the projectile will travel higher before returning to the ground, but the horizontal component of velocity will be unaffected.

So our sketch will consist of a parabola for the case $g = g'$ and a pair of parabolas joined at height h in the case $g' < g$. Note that the velocity does not change suddenly at height h, so the curve will be "smooth" at this point. (Technically, it has a continuous first derivative.)

We need to check that the particle does reach height h before we draw a sketch. The vertical component of velocity is initially $V \sin \alpha$. If the gravity were a constant g, then the maximum height reached would be s, where the formula $v^2 = u^2 + 2as$ gives us $0^2 = V^2 \sin^2 \alpha - 2gs$, so $s = V^2 \sin^2 \alpha / 2g > h$, so the particle does reach height greater than h.

So here, then, is the sketch:



Marks

M1 for checking that the particle reaches height h, either using "suvat" or energy considerations or otherwise.

A1 for correctly deducing that particle reaches height $> h$.

B1 for sketching parabola reaching height $> h$ in the case $g' = g$.

B1 for identical start to height h in the case $g' < g$. (NB: Cannot award this mark if two graphs drawn on separate axes.)

B1 for smooth joins at height h for the case $g' < g$.

B1 for reaching greater height and traversing greater distance in the case $g' < g$. (NB: If the two graphs are drawn on separate axes, can only award this mark if it is explicit that these distances are greater in the one case than in the other.)

[Total: 6 marks for this part]

Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'}\right) V \sin 2\alpha,$$

where $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

> ***Commentary:*** *We're going to have to take into account the impact of the change in gravity at height h. We must therefore split the trajectory into three parts: the initial part below height h, the part above height h, and the final part below height h, and use the standard "suvat" equations for each. However, we can save ourselves some work by noting that the trajectory is symmetrical, so we need only find the horizontal distance travelled to reach the highest point and then double it.*

We note that the horizontal speed is a constant $V \cos\alpha$, as there is no horizontal component of acceleration. Therefore the distance travelled is this times the time travelled. By symmetry, we find the time taken to reach the highest point on the trajectory, and then double it to find the total time.

First part: below height h.

We use the "suvat" equations to determine the time taken and the vertical speed at height h. Taking upwards as positive, we have $s = h$, $u = V\sin\alpha$, $a = -g$. Then $v^2 = u^2 + 2as$ gives $v^2 = V^2 \sin^2\alpha - 2gh$ and $v = u + at$ gives

$$\begin{aligned} t &= \frac{V \sin\alpha - \sqrt{V^2 \sin^2\alpha - 2gh}}{g} \\ &= \left(\frac{V - \sqrt{V^2 - 2gh \operatorname{cosec}^2\alpha}}{g}\right) \sin\alpha \\ &= \left(\frac{V - V'}{g}\right) \sin\alpha, \end{aligned}$$

writing $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

Second part: above height h.

This time, $u = \sqrt{V^2 \sin^2\alpha - 2gh} = V' \sin\alpha$, $v = 0$ and $a = -g'$, so $v = u + at$ gives

$$t = \frac{V' \sin\alpha}{g'}.$$

Therefore, the total time taken to reach the highest point is

$$\left(\frac{V - V'}{g}\right) \sin\alpha + \frac{V' \sin\alpha}{g'} = \left(\frac{V - V'}{g} + \frac{V'}{g'}\right) \sin\alpha.$$

Finally, we need to multiply this by 2 to get the total time taken and then by $V \cos\alpha$ to get the distance travelled, giving the distance

$$d = 2\left(\frac{V-V'}{g} + \frac{V'}{g'}\right)\sin\alpha . V\cos\alpha$$
$$= \left(\frac{V-V'}{g} + \frac{V'}{g'}\right) V\sin 2\alpha,$$

using $2\sin\alpha\cos\alpha = \sin 2\alpha$.

Marks

B1 for horizontal component of velocity is $V\cos\alpha$.

M1 for attempting to find time to height h.

M1 for attempting to find vertical component of velocity at height h.

A1 cao for finding v^2 or v at height h (vertical component).

A1 cao for finding time to height h.

M1 for using "suvat" or otherwise to find time from height h to greatest height or to return to height h.

A1 (ft) for time to greatest height or return to height h.

M1 for multiplying by two to get total time or calculating downward journey time.

A1 (ft) for total time to return to ground.

M1 for multiply time by horizontal component of velocity ($V\cos\alpha$).

A1 for finding distance (any form).

M1 for simplifying using V' (can give this mark anywhere in the question).

M1 for applying double angle formula.

A1 cso for deducing given expression.

[Total: 14 marks for this part]

Commentary: *Symmetry is powerful! Use it wherever you possibly can!*

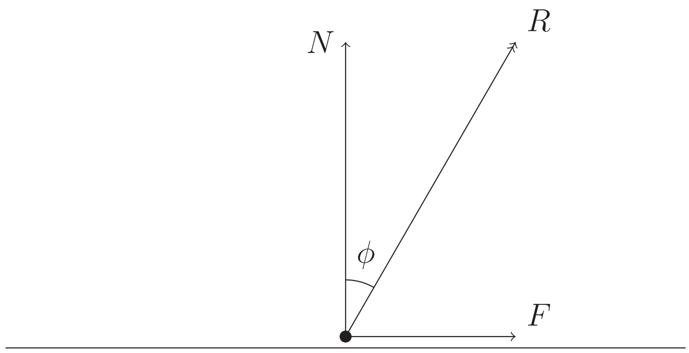
Question 11

A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ. Given that the rod is in limiting equilibrium (i.e., on the point of slipping at both ends), show that

$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2},$$

where α is the angle between P_1O and the vertical ($0 < \alpha < 45°$).
Let θ be the acute angle between the rod and the horizontal. Show that $\theta = 2\lambda$, where λ is defined by $\tan\lambda = \mu$ and $0 < \lambda < 22.5°$.

Commentary: *There are two ways of thinking about friction. One can talk about two separate reaction forces at a point of contact: the normal reaction N and the friction F, which are related by the equation $F \leqslant \mu N$, where μ is the coefficient of friction. Alternatively, one can consider the total reaction force, consisting of the resultant of the normal reaction and friction. This will be a force at an angle ϕ to the normal, where $\tan\phi = F/N$ (see diagram). This angle is known as the* angle of friction, *and we have $\tan\phi \leqslant \mu$. In both formulations, we have equality when the object is at the point of slipping.*



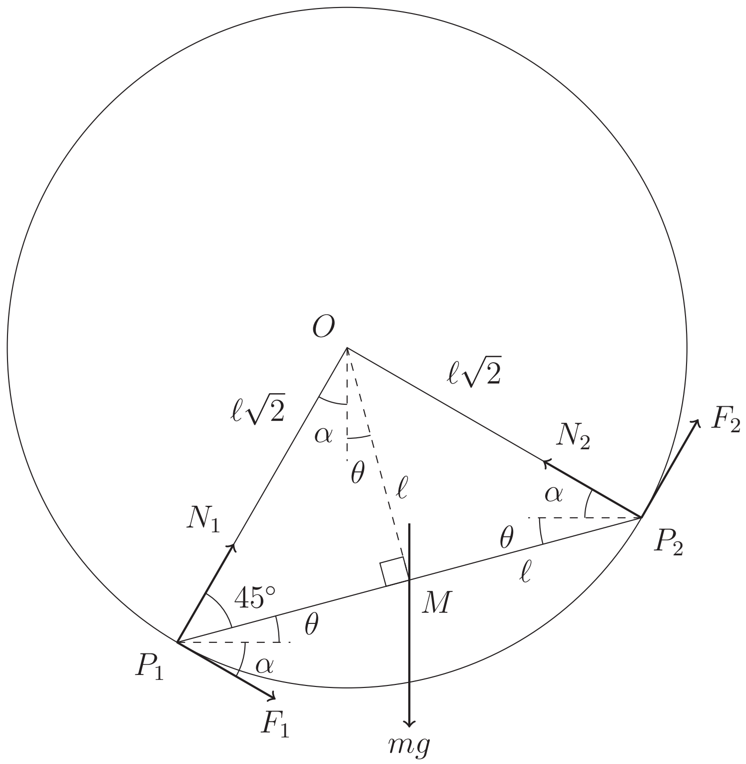
In this problem, we have a large body with only three forces on it: two total reaction forces, as we've just described, and one weight. It is a significant result that if a large body is in equilibrium and has exactly three forces acting on it, then the forces are either all parallel or all act through a single point. (This can be shown by taking moments about the point of intersection of two of the forces: the third force has to go through the same point if it is to be in equilibrium.)

We can use these ideas to our advantage in this case, solving the problem geometrically.

We present two methods for solving the problem. The first is a standard method using resolution of forces; the second is to use standard results about three forces acting on a large body. We specify that the length of the rod is 2ℓ, so that the radius of the circle is $\ell\sqrt{2}$.

Method 1: Resolving all the forces

We begin by drawing a clear sketch of the situation.



Note that as the rod is in limiting equilibrium, both of the frictional forces act to prevent it from slipping towards the horizontal, and $F_1 = \mu N_1$, $F_2 = \mu N_2$. Also, we see that $\theta = 45° - \alpha$, as the angle OP_1P_2 is 45° (the triangle being isosceles), so $0 < \theta < 45°$.

We now resolve the forces in two directions. We could resolve in any two directions, but so that we can exclude mg from at least one of the equations, we choose to resolve horizontally and vertically. Another sensible choice would have been to resolve along the directions of OP_1 and OP_2.

$$\mathscr{R}(\uparrow) \qquad N_1 \cos\alpha - \mu N_1 \sin\alpha + N_2 \sin\alpha + \mu N_2 \cos\alpha - mg = 0 \tag{1}$$
$$\mathscr{R}(\rightarrow) \qquad N_1 \sin\alpha + \mu N_1 \cos\alpha - N_2 \cos\alpha + \mu N_2 \sin\alpha = 0 \tag{2}$$

We also need to take moments. There are four obvious places about which we can take moments: O, P_1, P_2 and M. For completeness, we show what happens if we calculate moments about all four points; clearly only one of these is necessary.

$$\mathscr{M}(\widehat{O}) \qquad mg.\ell \sin(45° - \alpha) - F_1.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \tag{3}$$
$$\mathscr{M}(\widehat{P_1}) \qquad mg.\ell \cos(45° - \alpha) - N_2.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \tag{4}$$
$$\mathscr{M}(\widehat{P_2}) \qquad N_1.\ell\sqrt{2} - F_1.\ell\sqrt{2} - mg.\ell \cos(45° - \alpha) = 0 \tag{5}$$
$$\mathscr{M}(\widehat{M}) \qquad N_1.\ell/\sqrt{2} - F_1.\ell/\sqrt{2} - N_2.\ell/\sqrt{2} - F_2.\ell/\sqrt{2} = 0 \tag{6}$$

Our task is now to eliminate everything to find an expression for $\tan\alpha$ in terms of μ. We can use any one of the equations (3)–(6) to do this, but (6) appears to be the easiest to work with. (With the others, we would have to use a compound angle formula such as $\sin(45° - \alpha) = \sin 45° \cos\alpha - \cos 45° \sin\alpha = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$.)

Recalling that $F_1 = \mu N_1$ and $F_2 = \mu N_2$, equation (6) then gives us

$$N_1 - \mu N_1 = N_2 + \mu N_2$$

so

$$N_1(1-\mu) = N_2(1+\mu), \tag{7}$$

or equivalently

$$\frac{N_1}{N_2} = \frac{1+\mu}{1-\mu}. \tag{8}$$

$$\tag{9}$$

We can also solve equations (1) and (2) simultaneously to get the results

$$N_1 = \frac{mg(\cos\alpha - \mu\sin\alpha)}{1+\mu^2}$$

and

$$N_2 = \frac{mg(\sin\alpha + \mu\cos\alpha)}{1+\mu^2},$$

so

$$\frac{N_1}{N_2} = \frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha}.$$

Equating this expression with equation (8) yields

$$\frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha} = \frac{1+\mu}{1-\mu}.$$

Dividing the numerator and denominator of the left hand side by $\cos\alpha$ then gives

$$\frac{1-\mu\tan\alpha}{\tan\alpha + \mu} = \frac{1+\mu}{1-\mu}.$$

A simple rearrangement of this then yields our desired result:

$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}.$$

(Alternatively, one could use (7) to substitute for N_2 in equation (2), after writing $F_1 = \mu N_1$ and $F_2 = \mu N_2$. Factorising then gives

$$N_1(1+\mu)(\sin\alpha + \mu\cos\alpha) - N_1(1-\mu)(\cos\alpha - \mu\sin\alpha) = 0.$$

On dividing by $N_1\cos\alpha$ and expanding the brackets, we end up with an expression in $\tan\alpha$ which we can again rearrange to reach our desired conclusion.)

Now if $\tan\lambda = \mu$ with $0 < \lambda < 22.5°$, we have (recalling that $\theta = 45° - \alpha$)

$$\begin{aligned}
\tan\theta &= \tan(45° - \alpha) \\
&= \frac{\tan 45° - \tan\alpha}{1 + \tan 45° \tan\alpha} \\
&= \frac{1 - \tan\alpha}{1 + \tan\alpha} \qquad \text{as } \tan 45° = 1 \\
&= \frac{1 - \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}}{1 + \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}} \\
&= \frac{(1+2\mu-\mu^2) - (1-2\mu-\mu^2)}{(1+2\mu-\mu^2) + (1-2\mu-\mu^2)} \\
&= \frac{4\mu}{2-2\mu^2} \\
&= \frac{2\mu}{1-\mu^2} \\
&= \frac{2\tan\lambda}{1-\tan^2\lambda} \\
&= \tan 2\lambda.
\end{aligned}$$

Then since $0 < \theta < 45°$, it follows that $\theta = 2\lambda$ as required.

Marks

M1 for drawing a clear diagram with all five forces marked on correctly.

M1 A1 for resolving in one direction.

M1 A1 for taking moments about a point.

M1 A1 for resolving in another direction or taking moments around a different point.

B1 for using $F = \mu R$ (equality) somewhere in answer.

M1 A1 for finding an expression for N_1.

M1 A1 for finding an expression for N_2.

M1 A1 for substituting into third equation and finding an expression involving only $\sin\alpha$, $\cos\alpha$ (or $\tan\alpha$) and μ.

A1 cso for rearranging to find the stated expression for $\tan\alpha$.

[Total: 15 marks for this part]

B1 for stating $\theta = 45° - \alpha$.

M1 for expanding $\tan(45° - \alpha)$.

M1 for substituting in result from first part and simplifying resulting expression.

M1 for recognising double angle formula for $\tan 2\lambda$.

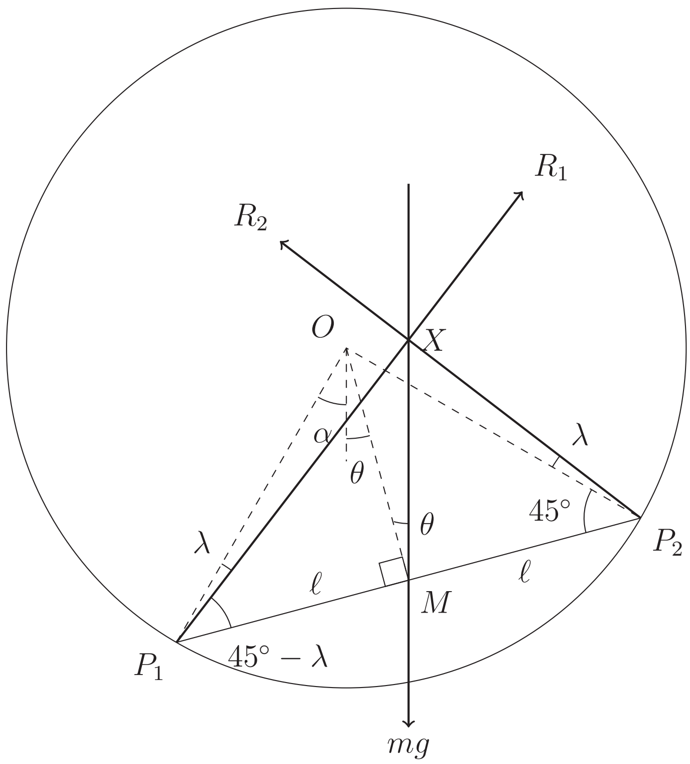
A1 cso for deducing $\theta = 2\lambda$.

[Total: 5 marks for this part]

Method 2: Three forces on a large body theorem

We recall the theorem that if three forces act on a large body in equilibrium, and they are not all parallel, then they are concurrent (i.e., they all pass through a single point). We combine the normal reaction, N, and friction, F, at each end of the rod into a single reaction force, R. This acts at an angle ϕ to the normal, where $\tan\phi = F/N$. In our case, $F = \mu N$ as the rod is on the point of slipping, so $\tan\phi = \mu$. But the question defines λ to be the acute angle such that $\tan\lambda = \mu$, from which it follows that $\phi = \mu$.

We now redraw the diagram showing only the total reaction forces, which we call R_1 and R_2 in this context, and we ensure that R_1, R_2 and the weight mg pass through a single point X.



The rest of the question is now pure trigonometry. We apply the sine rule to the triangles P_1MX and P_2MX, first noting that

$$\angle MXP_1 = 180° - (45° - \lambda) - (90° + \lambda) = 45° - \theta + \lambda$$

and

$$\angle MXP_2 = 180° - (90° - \theta) - (45° + \lambda) = 45° + \theta - \lambda$$

to get

$$\frac{\ell}{\sin(45° - \theta + \lambda)} = \frac{MX}{\sin(45° - \lambda)} \qquad (10)$$

$$\frac{\ell}{\sin(45° + \theta - \lambda)} = \frac{MX}{\sin(45° + \lambda)}\,. \qquad (11)$$

Then using the identity $\sin(90° - x) = \cos x$ twice, once with $x = 45° - \theta + \lambda$ and then with $x = 45° + \lambda$, we deduce from equation (11) that

$$\frac{\ell}{\cos(45° - \theta + \lambda)} = \frac{MX}{\cos(45° - \lambda)}\,. \qquad (12)$$

We can now divide equation (12) by (10) to get

$$\tan(45^\circ - \theta + \lambda) = \tan(45^\circ - \lambda). \tag{13}$$

It follows immediately that $45^\circ - \theta + \lambda = 45^\circ - \lambda$ as both angles are acute, so $\theta = 2\lambda$, which answers the second part of the question.

This result then leads us to conclude that

$$\tan\theta = \tan 2\lambda = \frac{2\tan\lambda}{1-\tan^2\lambda} = \frac{2\mu}{1-\mu^2}.$$

Finally, as we know that $\alpha = 45^\circ - \theta$, we can deduce that

$$\begin{aligned}\tan\alpha &= \tan(45^\circ - \theta)\\ &= \frac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta}\\ &= \frac{1-\tan\theta}{1+\tan\theta} \qquad \text{as } \tan 45^\circ = 1\\ &= \frac{1-\frac{2\mu}{1-\mu^2}}{1+\frac{2\mu}{1-\mu^2}}\\ &= \frac{(1-\mu^2)-2\mu}{(1-\mu^2)+2\mu}\\ &= \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2},\end{aligned}$$

and we are done.

Marks

B1 for quoting the three forces theorem correctly.

B1 for noting that we have the equality $F = \mu N$ as limiting equilibrium.

M1 A1 for deducing that $\phi = \lambda$.

M1 A1 for accurately sketching the situation (three forces concurrent) and angles correctly marked.

M1 A1 for calculating the angles $\angle P_1XM$ and $\angle P_2XM$.

M1 A1 for appropriately applying the sine rule to the triangles P_1XM and P_2XM.

M1 for using the identity $\sin(90^\circ - x) = \cos x$.

M1 A1 for deducing $\tan(45^\circ - \theta + \lambda) = \tan(45^\circ - \lambda)$ or similar.

M1 A1 for deducing $\theta = 2\lambda$.

[Total: 15 marks for this part]

B1 for stating that $\alpha = 45^\circ - \theta$.

M1 for applying the compound angle formula to $\tan(45^\circ - \alpha)$.

M1 for using the double angle formula to express $\tan 2\lambda$ in terms of μ.

M1 for simplifying resulting expression.

A1 cso for deducinmg the stated formula for $\tan\alpha$.

[Total: 5 marks for this part]

Commentary: *I personally prefer the second method—it is much more straightforward and the crucial result about three forces on a large body is a generally useful technique. Also, it does not require the calculation of the magnitudes of any of the forces, although it could certainly be done if desired.*

Question 12

In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \textit{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

The independent random variables X_1 and X_2 each take values 1, 2, …, N, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geqslant X_2 \\ X_2 & \text{if } X_2 \geqslant X_1. \end{cases}$$

(i) *Show that* $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ *for* $r = 1, 2, \ldots, N$.

Commentary: *In a question like this, it's worth drawing out a quick sample space diagram for a small case, say $N = 3$ or $N = 4$, to figure out exactly what is going on before getting stuck in. In the $N = 4$ case, we have*

		X_2			
		1	2	3	4
	1	1	2	3	4
X_1	2	2	2	3	4
	3	3	3	3	4
	4	4	4	4	4

We have $X = r$ when either $X_1 = r$ and $X_2 < r$, or $X_2 = r$ and $X_1 < r$ or $X_1 = X_2 = r$. Therefore

$$\begin{aligned}\mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 < r) + \mathrm{P}(X_2 = r \cap X_1 < r) + \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}.\end{aligned}$$

Alternatively, one can argue as follows:

$$\begin{aligned}\mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 \leqslant r) + \mathrm{P}(X_2 = r \cap X_1 \leqslant r) - \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r}{N} + \frac{1}{N}.\frac{r}{N} - \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}.\end{aligned}$$

Marks

M1 for expressing event $X = r$ in terms of X_1 and X_2.

M1 for evaluating a probability of the form $P(X_1 = r \cap X_2 < r)$ or similar.
A1 cso for deriving correct answer as given.
[Total: 3 marks for part (i)]

(ii) *Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.*

Commentary: *The explicit example allows us to check our algebra, which is always valuable.*

By the definition of expectation, we have

$$\begin{aligned}
\mu = \mathrm{E}(X) &= \sum_{r=1}^{N} r.\mathrm{P}(X = r) \\
&= \sum_{r=1}^{N} \frac{r(2r-1)}{N^2} \\
&= \frac{1}{N^2} \sum_{r=1}^{N} (2r^2 - r) \\
&= \frac{1}{N^2} \big(\tfrac{2}{6}N(N+1)(2N+1) - \tfrac{1}{2}N(N+1)\big) \quad \text{using the given results} \\
&= \frac{\tfrac{1}{6}N(N+1)(4N+2-3)}{N^2} \\
&= \frac{(N+1)(4N-1)}{6N}\,.
\end{aligned}$$

In the case $N = 100$, this is $101 \times 399/600 = 13\,433/200 = 67.165$ as required.

Marks
M1 for applying the definition of expectation with the result from (i).
M1 for applying the stated results for $\sum r$ and $\sum r^2$.
M1 for simplifying the resulting algebra.
A1 cao or equivalent form.
A1 for correctly substituting $N = 100$ to obtain the stated result.
[Total: 5 marks for part (ii)]

(iii) *The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. Find an expression for m in terms of N and give an explicit value for m in the case $N = 100$.*

We have

$$\begin{aligned}
\mathrm{P}(X \leqslant k) &= \sum_{r=1}^{k} \frac{2r-1}{N^2} \\
&= \frac{1}{N^2}\big(2.\tfrac{1}{2}k(k+1) - k\big) \\
&= \frac{k^2}{N^2}
\end{aligned}$$

so that

$$\mathrm{P}(X \geqslant k) = 1 - \mathrm{P}(X \leqslant k-1) = 1 - \frac{(k-1)^2}{N^2}\,.$$

We are looking for the value of m which makes $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. The first condition gives

$$\begin{aligned}
& 1 - \frac{(m-1)^2}{N^2} \geqslant \tfrac{1}{2} \\
\iff \quad & \frac{(m-1)^2}{N^2} \leqslant \tfrac{1}{2} \\
\iff \quad & (m-1)^2 \leqslant \tfrac{1}{2}N^2 \\
\iff \quad & m-1 \leqslant \tfrac{N}{\sqrt{2}} \\
\iff \quad & m \leqslant \tfrac{N}{\sqrt{2}} + 1.
\end{aligned}$$

The second condition, $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$, yields

$$\begin{aligned}
& \frac{m^2}{N^2} \geqslant \tfrac{1}{2} \\
\iff \quad & m^2 \geqslant \tfrac{1}{2}N^2 \\
\iff \quad & m \geqslant \tfrac{N}{\sqrt{2}}\,.
\end{aligned}$$

So we have $N/\sqrt{2} \leqslant m \leqslant (N/\sqrt{2}) + 1$, thus m is the smallest integer greater than $N/\sqrt{2}$ (which is not itself an integer as $\sqrt{2}$ is irrational). The smallest integer greater than or equal to a number x is called the *ceiling* of x, and is denoted by $\lceil x \rceil$, so we can state our result as $m = \lceil N/\sqrt{2} \rceil$.

In the case $N = 100$, $m = \lceil 100/\sqrt{2} \rceil = \lceil 70.7\ldots \rceil = 71$.

> ***Commentary:*** *If you don't happen to know that* $\sqrt{2} = 0.707\ldots$, *then you can calculate as follows:* $100/\sqrt{2} = \sqrt{10\,000/2} = \sqrt{5000}$. *Now* $70^2 = 4900$ *and* $71^2 = 5041$, *so* $70 < 100/\sqrt{2} < 71$.

Marks

M1 A1 for calculating $P(X \leqslant k)$.
A1 (ft) for finding $P(X \geqslant k)$.
M1 for simplifying $P(X \leqslant m) \geqslant \frac{1}{2}$.
M1 for simplifying $P(X \geqslant m) \geqslant \frac{1}{2}$.

M1 A1 for deducing that $m = \lceil N/\sqrt{2} \rceil$ or equivalent.
A1 for deducing $m = 71$ in the case $N = 100$.
[Total: 8 marks for part (iii)]

(iv) *Show that when N is very large,*

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}.$$

We have formulæ for μ from part (ii) and for m from part (iii). Therefore we have, for large N,

$$\begin{aligned}
\frac{\mu}{m} &= \frac{(N+1)(4N-1)}{6N} \Big/ \lceil N/\sqrt{2} \rceil \\
&\approx \frac{(N+1)(4N-1)}{6N(N/\sqrt{2})} \\
&= \frac{4N^2+3N-1}{6N^2/\sqrt{2}} \\
&= \frac{4+\frac{3}{N}-\frac{1}{N^2}}{6/\sqrt{2}} \\
&\approx \frac{4}{6/\sqrt{2}} \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}$$

Marks
M1 for substituting formulæ from above into μ/m.
M1 for approximating $m \approx N/\sqrt{2}$ (no justification needed).
M1 for dividing numerator and denominator by N^2 or similar argument.
A1 for deducing stated result.
[Total: 4 marks for part (iv)]

Question 13

Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) *Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.*

Commentary: *Always label the people in situations such as this. Either 'M' and 'W' for 'Man' and 'Woman' or 'H' and 'W' for 'Husband' and 'Wife'. And draw pictures to make sure that you know what is going on! In this case, the table is round, so any position can be rotated as much as we like without changing anything. So we might as well fix the position of the first husband. There are $5! = 120$ possible arrangements for the remaining five people. One way to do this question—not the fastest or neatest, but a possible way to check your answers—is to list all of these possibilities and note which ones satisfy the conditions. There are several ways to tackle this question besides that given here.*

We call the couples H_1 and W_1, H_2 and W_2, H_3 and W_3. We seat H_1 arbitrarily, leaving $5! = 120$ ways of seating the remaining five people. If each husband sits next to his wife, then there are two seats in which W_1 can sit, each of which leaves four consecutive seats for the other two couples.

There are four choices for who to sit immediately next to W_1, and that forces the following seat as well (being the spouse of that person).

Next, there are two choices for who to seat the other side of H_1, and the final person must sit next to their spouse.

So there are $2 \times 4 \times 2 = 16$ ways to have all of the husbands sitting next to their wives, with a probability of $\frac{16}{120} = \frac{2}{15}$.

Marks

M1 for labelling each person uniquely.

M1 for noting that we can sit one specified person in a fixed seat with $5!$ ways for the rest, or for explicitly working with all $6!$ possibilities.

M1 for two ways to sit W_1 (with fixed H_1) or similar argument.

M1 for determining the possibilities for the remaining people.

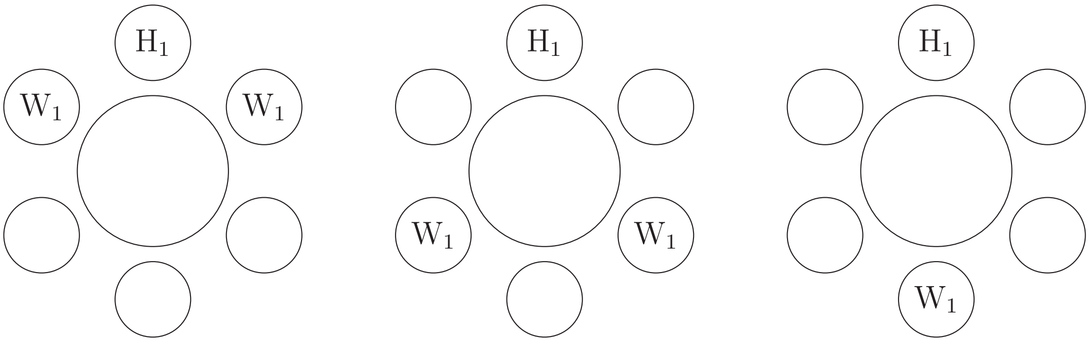
A1 cso for reaching the stated result.

[Total: 5 marks for part (i)]

(ii) *Find the probability that exactly two husbands sit next to their wives.*

Commentary: *Listing cases as we did for part (i) is going to get very messy very quickly. So we might do better to draw a picture of all of the possible cases.*

Assume to begin with that H_1 and W_1 are separated and the other two husbands are seated next to their wives. Once H_1 has been seated, there are five possible positions for W_1, as shown in the following diagrams (there are two possibilities shown in each of the first two):



Clearly the first two possibilities do not work: in the first, H_1 and W_1 are next to each other. In the second, whoever sits between H_1 and W_1 will be separated from their spouse. So W_1 must sit opposite H_1, one couple sits to the right of H_1 and the other couple to his left. There are two choices for which couple sits to the right of H_1, and two choices for whether the husband or wife sits next to H_1; similarly there are two choices for whether the husband or wife of the third couple sits next to H_1. So in all, there are $2 \times 2 \times 2 = 8$ ways to seat the couples with H_1 and W_1 separated and the other couples together.

Similarly, there are 8 ways with couple 2 separated and 8 ways with couple 3 separated, so there are $3 \times 8 = 24$ ways in total.

Thus the probability is $\frac{24}{120} = \frac{3}{15} = \frac{1}{5}$.

> ***Marks***
> *M1 for choosing who to separate.*
> *M1 for considering possible seating configurations.*
> *M1 for arguing which configurations are valid in this case.*
> *M1 for counting the possible arrangements with* H_1 *and* W_1 *separated.*
> *M1 for multiplying by 3 to take into account which couple are separated.*
> *A1 cao for the correct probability.*
> *[Total: 6 marks for this part]*

(iii) *Find the probability that no husband sits next to his wife.*

> ***Commentary:*** *There are two approaches to this: we can either calculate the probability of exactly one husband sitting next to his wife and then combine this with the answers to (i) and (ii) to determine the probability of at least one husband sitting next to his wife. Alternatively, we can calculate this directly.*

Method 1: First find the probability of exactly one husband sitting next to his wife.

Let us assume that H_3 and W_3 are the only pair next to each other. Then in the above diagrams, the left hand one fails as H_1 and W_1 are together. The right hand one also fails, as if H_3 and W_3 are together, H_2 and W_2 must also be. So the only valid configuration is the middle one, with either H_2 or W_2 between H_1 and W_1 and the other partner on the other side of either H_1 or W_1.

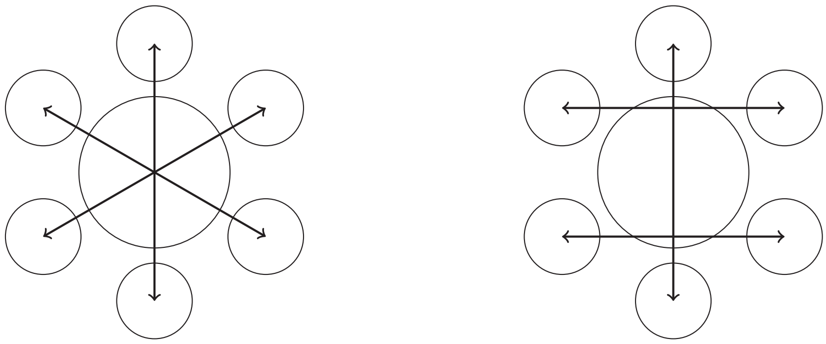
There are two choices for where W_1 will sit, two choices for which of H_2 or W_2 will sit between H_1 and W_1, two choices for where the other partner will sit, and two choices for which way round H_3 and W_3 will sit, giving $2 \times 2 \times 2 \times 2 = 16$ possibilities.

Finally, we need to multiply this by three, as any one of the three couples could be the adjacent one, giving $3 \times 16 = 48$ possibilities, and hence a probability of $\frac{48}{120} = \frac{2}{5} = \frac{6}{15}$.

Thus the probability that no husband sits next to his wife is $1 - \frac{2}{15} - \frac{3}{15} - \frac{6}{15} = \frac{4}{15}$.

Method 2: Find the probability directly.

If no husband sits next to his wife, there are two possible configurations as shown in these diagrams (the arrows join husbands with their wives):



The number of ways of arranging the first case (fixing H_1 at the top as usual) is $2 \times 2 \times 2 = 8$, as their are two ways of choosing which couple sits in which of the diagonal pairs of seats, two ways of couple 2 sitting and two ways of couple 3 sitting.

For the second case, which is no longer totally symmetrical between the three couples, if H_1 sits in the top seat, there are again 8 ways of seating the other two couples. As there are three choices for which couple sits opposite each other, there are $3 \times 8 = 24$ ways in all.

Thus in total there are $8 + 24 = 32$ ways, giving a total probability of $\frac{32}{120} = \frac{4}{15}$.

Marks

Method 1:

M1 for using $\mathrm{P}(\textit{none}) = 1 - \mathrm{P}(\geqslant 1)$.

M1 for specifying which couple together.

M1 A1 for deducing the possible configurations.

M1 A1 for counting the possible arrangements with the specified couple together.

M1 for multiplying this by 3.

A1 cao for the probability of exactly one couple together.

A1 (ft) for finding probability of no couples together.

Method 2:

M1 A2 for determining valid configurations (A1 for each configuration; deduct one A mark for every invalid configuration included to a minimum of A0).

M1 A1 for counting possibilities for the left hand configuration.

M1 A1 for counting possibilities for the right hand configuration with a specified couple opposite each other.

M1 for multiplying the latter answer by 3.

A1 cao for the probability of no couples together.

[Total: 9 marks for part (iii)]