STEP I 2010 Solutions and Mark Scheme

June 2010

Contents

Foreword

This document contains model solutions to the 2010 STEP Mathematics Paper I. The solutions are fully worked and contain more detail and explanation than would be expected from candidates. They are intended to help students understand how to answer the questions, and therefore they are encouraged to attempt them first before looking at these model answers.

This document also contains a Mark Scheme. This was used by the examiners during the marking process. It is important to remember that the nature of these questions is such that there may be multiple acceptable ways of answering them. As in any examination, the mark scheme was adapted appropriately for these alternative approaches; these adaptations are not recorded here.

The meanings of the marks are as in the standard GCSE and AS/A2 mark schemes:

- M marks for method
- A marks for correct answers, dependent on gaining the corresponding M mark(s)
- B marks are independent accuracy marks
- ft means that incorrect working is followed through
- dep means this method mark is dependent upon gaining the previous method mark
- cao/cso means 'correct answer/solution only'
- SC means 'special case', and applies when the regular mark scheme has given the student (almost) no marks
- 'condone . . . ' means 'award the mark even if the candidate has made the specified error'

Question 1

Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d,$$

find the values of the constants a, b, c and d.

We expand the right hand side, and then equate coefficients:

$$\begin{aligned}
5x^2 + 2y^2 - 6xy + 4x - 4y& \\
&\equiv a(x - y + 2)^2 + b(cx + y)^2 + d \\
&\equiv a(x^2 - 2xy + y^2 + 4x - 4y + 4) + b(c^2x^2 + 2cxy + y^2) + d \\
&\equiv (a + bc^2)x^2 + (2bc - 2a)xy + (a + b)y^2 + 4ax - 4ay + 4a + d,
\end{aligned}$$

so we require

$$\begin{aligned}
a + bc^2 &= 5 \\
2bc - 2a &= -6 \\
a + b &= 2 \\
4a &= 4 \\
-4a &= -4 \\
4a + d &= 0.
\end{aligned}$$

The fourth and fifth equations both give $a = 1$ immediately, giving $b = 1$ from the third equation. Then the second equation gives $c = -2$ and the final equation gives $d = -4$.

We must also check that this solution is consistent with the first equation. We have $a + bc^2 = 1 + 1 \times (-2)^2 = 5$, as required. (Why is this necessary? Well, if the second equation had begun with $7x^2 + \cdots$, then our method would still have given us $a = 1$, etc., but the coefficients for the x^2 term would not have matched, so we would not have been able to write the second equation in the same way as the first.)

We thus deduce that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv (x - y + 2)^2 + (-2x + y)^2 - 4.$$

Marks

M1: Expanding at least one of the squares on the right hand side

M1: Fully expanding the right side; condone at most two algebraic errors for this mark

M1: Collecting x^2, etc., terms correctly (follow through errors from first steps)

M1: Equate coefficients of all x^2, etc., terms to determine equations for a, b, etc.

A1: Sufficient correct equations to find a, b, c, d

M1: Reasonable attempt at solving resulting equations

A1 cao: All four of a, b, c, d correct from correct working

[Total for this part: 7 marks]

> *Solve the simultaneous equations*
>
> $$5x^2 + 2y^2 - 6xy + 4x - 4y = 9, \tag{1}$$
> $$6x^2 + 3y^2 - 8xy + 8x - 8y = 14. \tag{2}$$

Spurred on by our success in the first part, we will rewrite the first equation in the suggested form:

$$(x - y + 2)^2 + (y - 2x)^2 - 4 = 9. \tag{3}$$

We are led to wonder whether the same trick will work for the second equation, so let's try writing:

$$6x^2 + 3y^2 - 8xy + 8x - 8y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d.$$

As before, we get equations:

$$\begin{aligned} a + bc^2 &= 6 \\ 2bc - 2a &= -8 \\ a + b &= 3 \\ 4a &= 8 \\ -4a &= -8 \\ 4a + d &= 0. \end{aligned}$$

(We can write these down as the right hand side is the same as before.)

This time, $a = 2$ from both the fourth and fifth equations, so we get $b = 1$ from the third equation. The second equation gives us $c = -2$. Finally, the sixth equation gives us $d = -8$.

We *must* now check that our solution is consistent with the first equation, which we have not yet used. The left hand side is $a + bc^2 = 2 + 1 \times (-2)^2 = 6$, which works, so we can write the second equation as

$$2(x - y + 2)^2 + (y - 2x)^2 - 8 = 14.$$

(If we had not checked for consistency, we might have wrongly concluded that $183x^2 + 3y^2 - 8xy + 8x - 8y$ can also be written in the same way.)

These two equations now look remarkably similar! In fact, let's move the constants to the right hand side and write them together:

$$\begin{aligned} (x - y + 2)^2 + (y - 2x)^2 &= 13 \\ 2(x - y + 2)^2 + (y - 2x)^2 &= 22. \end{aligned}$$

We now have two simultaneous equations which look almost linear. In fact, if we write $u = (x - y + 2)^2$ and $v = (y - 2x)^2$, we get

$$\begin{aligned} u + v &= 13 \\ 2u + v &= 22 \end{aligned}$$

which we can easily solve to get $u = 9$ and $v = 4$.

Therefore, we now have to solve the two equations

$$(x - y + 2)^2 = 9 \tag{4}$$
$$(y - 2x)^2 = 4. \tag{5}$$

We can take square roots, so that (4) gives $x - y + 2 = \pm 3$ and (5) gives $y - 2x = \pm 2$.

Thus we now have four possibilities (two from equation (4), and for each of these, two from equation (5)), and we solve each one, checking our results back in the original equations.

$2x - y$	$x - y + 2$	x	y	LHS of (1)	LHS of (2)
2	3	1	0	9	14
2	−3	7	12	9	14
−2	3	−3	−4	9	14
−2	−3	3	8	9	14

Therefore we see that the four solutions are $(x, y) = (1, 0)$, $(7, 12)$, $(-3, -4)$ and $(3, 8)$.

An alternative is to observe that equation (2) looks almost double equation (1), so we consider $2 \times (1) - (2)$:

$$4x^2 + y^2 - 4xy = 4.$$

But the left hand side is simply $(2x - y)^2$, so we get $2x - y = \pm 2$.

Substituting this into equation (3) gives us

$$(x - y + 2)^2 + 4 - 4 = 9,$$

so that $x - y + 2 = \pm 3$.

Thus we have the four possibilities we found in the first approach, and we continue as above.

Yet another alternative approach is to subtract $(2) - (1)$ to get

$$x^2 + y^2 - 2xy + 4x - 4y = 5,$$

so that

$$(x - y)^2 + 4(x - y) = 5.$$

Writing $z = x - y$, we get the quadratic $z^2 + 4z - 5 = 0$, which we can then factorise to give $(z + 5)(z - 1) = 0$, so either $z = 1$ or $z = -5$, which gives $x - y = 1$ or $x - y = -5$.

Substituting $x - y = 1$ into (3) now gives

$$(1 + 2)^2 + (y - 2x)^2 - 4 = 9,$$

so that $(y - 2x)^2 = 4$; substituting $x - y = -5$, on the other hand, would lead us to

$$(-5 + 2)^2 + (y - 2x)^2 - 4 = 9,$$

and again we deduce $(y - 2x)^2 = 4$.

We have again reached the same deductions as in the first approach, so we continue from there.

Marks

First approach

M1: Attempting to rewrite the second equation as in the first part

M1: Setting up five or six equations as before (don't need to rework the expansion steps from first part)

M1: Solving the five or six equations to find a, b, c, d

A1 cao: All four of a, b, c, d correct

B1: Checking that the values found work in all six equations (sufficient to just say that they have checked them)

M1: Solving resulting equations simultaneously for $(x-y+2)^2$ and $(y-2x)^2$

A1: Doing it correctly

A1 cso: Finding both possible values for $x-y+2$

A1 cso: Finding both possible values for $y-2x$

M2: Working through all four possibilities for $2x-y$ and $x-y+2$ (M1 if only one or two possibilities considered)

NB: If more than four solutions found, some of the earlier marks must necessarily have been lost, and of the remaining marks, the maximum possible will be A1 B0 (A1 only if they find all four correct answers in addition to their spurious answer(s), and B0 as they cannot possibly have correctly checked their answers). The same comment applies to the remaining approaches.

A2 cao: Correctly deducing x and y for all four possibilities (A1 if at least one but fewer than four possibilities correctly determined, and A1 ft if earlier error but follows through at least two possibilities)

First alternative approach

M1: Performing a subtraction to eliminate linear terms

A1: Doing it correctly

M1: Factorising left hand side of resulting equation

A1: Doing it correctly

A1: Correctly taking square root to get $2x-y=\pm 2$ (need $\pm$ for this mark)

M1: Reasonable attempt to substitute into one of the original equations or the rearranged first equation ...

A2 ft: and doing it correctly, either A1 for each case, or A2 for substituting into simplified form ...

A1 ft: and solving this to find $x-y+2=\pm 3$

M2: Working through all four possibilities for $2x-y$ and $x-y+2$ (M1 if only one or two possibilities considered)

A2 cao: Correctly deducing x and y for all four possibilities (A1 if at least one but fewer than four possibilities correctly determined, and A1 ft if earlier error but follows through at least two possibilities)

Yet another alternative approach

M1: Subtracting the two equations to get $x^2 + y^2 + \cdots$

M1 dep: Factorising the resulting equation towards $(x-y)^2 + 4(x-y)$ or to $(x-y)(x-y+4)$

A1: Correct factorisation

M1: Substituting $z = x - y$ (can be implicit)

M1: Solving quadratic in z

A1: Correct possibilities for $x - y$

M1: Substituting each solution into (3)

A1: Deducing $(y-2x)^2 = 4$ for one case

A1: Deducing $(y-2x)^2 = 4$ for the other case

M2: Working through all four possibilities for $y - 2x$ and $x - y$ (M1 if only one or two possibilities considered)

A2 cao: Correctly deducing x and y for all four possibilities (A1 if at least two but fewer than four possibilities correctly determined, and A1 ft if earlier error but follows through at least two possibilities)

Second alternative approach

A further approach which some candidates might take runs as follows. After having deduced that $2x - y = \pm 2$, we can rearrange to get $y = 2x \pm 2$ and substitute this into (1) to get

$$5x^2 + 2(2x \pm 2)^2 - 6x(2x \pm 2) + 4x - 4(2x \pm 2) = 9$$

which we can simplify to

$$x^2 + (-4 \pm 4)x + (-1 \mp 8) = 0$$

(although candidates are more likely, if they get this far, to do two separate calculations). The two resulting quadratics, $x^2 - 9 = 0$ when $y = 2x + 2$ and $x^2 - 8x + 7$ when $y = 2x - 2$ immediately give our required solutions.

M1 A1 M1 A1 A1: First five marks as above, first alternative method, then:

M1: Writing as $y = 2x \pm 2$ or similar and substituting into either (1) or (2) to eliminate either y or x

M1: Splitting into two cases and expanding brackets in each

A1: Reaching at least one correct quadratic equation for x or y

M1: Solving resulting equations to determine x or y

A1: For finding either x or y correctly

M1: Substituting back into $y = 2x \pm 2$ or equivalent to determine the other unknown

A2: All four correct solution pairs (A1 if at least one, but fewer than four, correct)

[Total for this part: 13 marks]

Question 2

The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, *where* a *and* b *are constants, has two stationary points. Show that*

$$a - b < 0 \qquad \text{or} \qquad a - b > 4.$$

We begin by differentiating using first the product rule and then the quotient rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-a}{x-b}\right)\mathrm{e}^x + \frac{(x-a)}{(x-b)}\mathrm{e}^x \\
&= \frac{(x-b).1-(x-a).1}{(x-b)^2}\mathrm{e}^x + \frac{(x-a)}{(x-b)}\mathrm{e}^x \\
&= \frac{(a-b)}{(x-b)^2}\mathrm{e}^x + \frac{(x-a)(x-b)}{(x-b)^2}\mathrm{e}^x \\
&= \frac{x^2-(a+b)x+(ab+a-b)}{(x-b)^2}\mathrm{e}^x.
\end{aligned}$$

Now solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives $x^2-(a+b)x+(ab+a-b) = 0$. Since the curve has two stationary points, this quadratic must have two distinct real roots. Therefore the discriminant must be positive, that is

$$(a+b)^2 - 4(ab+a-b) > 0,$$

and expanding gives $a^2 - 2ab + b^2 - 4a + 4b > 0$, so $(a-b)^2 - 4(a-b) > 0$. Factorising this last expression gives

$$(a-b)(a-b-4) > 0,$$

so (sketching a graph to help, possibly also replacing $a-b$ with a variable like x), we see that we must either have $a-b<0$ or $a-b>4$.

Marks

M1: Using either the product or quotient rule to do one step of the differentiation
M1 dep: Using either the product or quotient rule to complete the differentiation
A1: Correct derivative
M1: Finding simplified quadratic condition for location stationary points. (Can award even if derivative is incorrect, as long as at least one of first two M marks was awarded.)
M1 dep: Determining discriminant condition for two stationary points
M1 dep: Simplifying and factorising determinant
A1 cso: Deducing stated condition for two stationary points
[Total for this part: 7 marks]

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

We are studying the curve $y = \left(\dfrac{x}{x - \frac{1}{2}}\right)\mathrm{e}^x$.

We have $a - b = -\frac{1}{2} < 0$, so the curve has two stationary points by the first part of the question. The x-coordinates of the stationary points are found by solving the quadratic

$$x^2 - (a+b)x + (ab + a - b) = 0,$$

as above.

Substituting in our values for a and b, we get $x^2 - \frac{1}{2}x - \frac{1}{2} = 0$, so $2x^2 - x - 1 = 0$, which factorises to $(x-1)(2x+1) = 0$. Thus there are stationary points at $(1, 2\mathrm{e})$ and $(-\frac{1}{2}, \frac{1}{2}\mathrm{e}^{-1/2})$.

The vertical asymptote is at $x = b$, that is at $x = \frac{1}{2}$.

Therefore, since the two stationary points are at $x = 1$ and $x = -\frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote.

We note that the only time the curve crosses the x-axis is when $x = a$, so this is when $x = 0$, and this is also the y-intercept in this case.

As $x \to \pm\infty$, $y \sim \mathrm{e}^x$ (meaning y is approximately equal to e^x; formally, we say that y is *asymptotically equal* to e^x), as the fraction $(x-a)/(x-b)$ tends to 1.

We can also note where the curve is positive and negative: since e^x is always positive, $y > 0$ whenever both $x - a > 0$ and $x - b > 0$, or when both $x - a < 0$ and $x - b < 0$, so $y < 0$ when x lies between a and b and is positive or zero otherwise.

Using all of this, we can now sketch the graph of the function. The nature of the stationary points will become clear from the graphs. In the graph, the dotted lines are the asymptotes ($x = \frac{1}{2}$ and $y = \mathrm{e}^x$) and the red line is the graph we want, with the stationary points indicated.

Marks

M1: Substituting a and b into earlier equation to find x-coordinates of stationary points (or equivalent method)

A1: Correct x-coordinates of both stationary points

B1: Vertical asymptote equation, and showing or stating that x-coordinates are both sides of asymptote (needs to be explicit, as question has asked them to show this)

A1 ft: Correct y-coordinates of both stationary points (if x-coordinates incorrect, then can only award this ft mark if they are either side of $x = \frac{1}{2}$)

B1: Correct general shape of graph (correct axis crossing and approaching vertical asymptote in correct directions, but not necessarily approaching e^x as $x \to \pm\infty$)

B1: Correct indications of stationary points and whether they are max/min points
B1: Description of asymptotic behaviour as $x \to \pm\infty$ (could be in writing or indicated on graph); sufficient to say $y \to 0$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$
[Total for this part: 7 marks]

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

This time, we are studying the curve $y = \left(\dfrac{x - \frac{9}{2}}{x}\right)\mathrm{e}^x$.

Proceeding as in (i), we have $a - b = \frac{9}{2} > 4$, so again, the curve has two stationary points. The x-coordinates of the stationary points are given by solving the quadratic

$$x^2 - (a+b)x + (ab + a - b) = 0,$$

as above.

Substituting our values, we get $x^2 - \frac{9}{2}x + \frac{9}{2}$, so $2x^2 - 9x + 9 = 0$. Again, this factorises nicely to $(x-3)(2x-3) = 0$, giving stationary points at $(\frac{3}{2}, -2\mathrm{e}^{3/2})$ and $(3, -\frac{1}{2}\mathrm{e}^3)$.

The vertical asymptote is at $x = b$, that is at $x = 0$. This time, therefore, the stationary points are both to the right of the vertical asymptote.

The x-intercept is at $x = a$, that is, at $(\frac{9}{2}, 0)$. There is no y-intercept as $x = 0$ is an asymptote.

Again, as $x \to \pm\infty$, $y \sim \mathrm{e}^x$.

As in (i), $y < 0$ when x lies between a and b and is positive or zero otherwise.

Using all of this, we can now sketch the graph of this function. Note that the asymptote $y = \mathrm{e}^x$ is much greater than y until x is greater than 20 or so, as even then $(x-a)/(x-b) \approx 15/20$, and only slowly approaches 1. We don't even attempt to sketch the function for such large values of x!

Marks
B1: Correct x-coordinates of both stationary points [No method marks here!]
B1 ft: Correct y-coordinates of both stationary points (if x-coordinates incorrect, then can only award this ft mark if x-coords are both positive)
B1: Correct vertical asymptote and axis crossing
B1: Correct indications of stationary points on sketch and whether they are max/min points
B1: Correct general shape of graph (approaching vertical asymptote in correct directions, but not necessarily approaching e^x as $x \to \pm\infty$)
B1: Description of asymptotic behaviour as $x \to \pm\infty$ (could be in writing or indicated on graph); again, sufficient to say $y \to 0$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$

[Total for this part: 6 marks]

Question 3

Show that

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

and deduce that

$$\sin A - \sin B = 2\cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$

We use the compound angle formulæ (also called the addition formulæ) to expand the left hand side, getting:

$$\begin{aligned}\sin(x+y) - \sin(x-y) &= (\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y) \\ &= 2\cos x \sin y,\end{aligned}$$

as required.

For the deduction, we want $A = x+y$ and $B = x-y$, so $x = \frac{1}{2}(A+B)$ and $y = \frac{1}{2}(A-B)$, solving these two equations simultaneously to find x and y. Then we simply substitute these values of x and y into our previous identity, and we reach the desired conclusion:

$$\sin A - \sin B = 2\cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$

(This identity is known as one of the *factor formulæ*.)

Marks
M1: Using the compound angle formula for either $\sin(x+y)$ *or* $\sin(x-y)$
A1: Correctly using both compound angle formulæ to deduce first identity
M1: Setting $A = x+y$ and $B = x-y$ in first identity (or setting $x = (A+B)/2$, $y = (A-B)/2$)
M1: Rearranging to get x and y in terms of A and B (or rearranging to get A and B in terms of x and y)
A1 cso: Deducing required identity
[Total for this part: 5 marks]

Show also that

$$\cos A - \cos B = -2\sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$

Likewise, we have

$$\begin{aligned}\cos(x+y) - \cos(x-y) &= (\cos x \cos y - \sin x \sin y) - (\cos x \cos y + \sin x \sin y) \\ &= -2\sin x \sin y,\end{aligned}$$

so again substituting $x = \frac{1}{2}(A+B)$ and $y = \frac{1}{2}(A-B)$ gives

$$\cos A - \cos B = -2\sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$

Marks

M1: Expanding $\cos(x+y) - \cos(x-y)$ *using compound angle formulæ*

M1: Using same substitution (A, B) *as for first identity*

A1 cso: Deducing required identity

[Total for this part: 3 marks]

The points P, Q, R *and* S *have coordinates* $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ *and* $(a\cos s, b\sin s)$ *respectively, where* $0 \leqslant p < q < r < s < 2\pi$, *and* a *and* b *are positive.*

Given that neither of the lines PQ *and* SR *is vertical, show that these lines are parallel if and only if*

$$r + s - p - q = 2\pi.$$

Remark: The points P, Q, R *and* S *all lie on an* ellipse, *which can be thought of as a stretched circle, as their coordinates all have* $\frac{x}{a} = \cos\theta$ *and* $\frac{y}{b} = \sin\theta$, *so they satisfy the equation* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

The lines PQ and SR are parallel if and only if their gradients are equal (and neither are vertical, so their gradients are well-defined), thus

$$\begin{aligned}
PQ \parallel RS &\iff \frac{b\sin q - b\sin p}{a\cos q - a\cos p} = \frac{b\sin s - b\sin r}{a\cos s - a\cos r} \\
&\iff \frac{\sin q - \sin p}{\cos q - \cos p} = \frac{\sin s - \sin r}{\cos s - \cos r} \\
&\iff \frac{2\cos\frac{1}{2}(q+p)\sin\frac{1}{2}(q-p)}{-2\sin\frac{1}{2}(q+p)\sin\frac{1}{2}(q-p)} = \frac{2\cos\frac{1}{2}(s+r)\sin\frac{1}{2}(s-r)}{-2\sin\frac{1}{2}(s+r)\sin\frac{1}{2}(s-r)} \\
&\iff \frac{\cos\frac{1}{2}(q+p)}{-\sin\frac{1}{2}(q+p)} = \frac{\cos\frac{1}{2}(s+r)}{-\sin\frac{1}{2}(s+r)} \\
&\iff \cot\tfrac{1}{2}(q+p) = \cot\tfrac{1}{2}(s+r) \\
&\iff \tfrac{1}{2}(q+p) = \tfrac{1}{2}(s+r) + k\pi \qquad \text{for some } k \in \mathbb{Z} \\
&\iff q + p = s + r + 2k\pi \qquad \text{for some } k \in \mathbb{Z} \\
&\iff r + s - p - q = 2n\pi \qquad \text{for some } n \in \mathbb{Z}.
\end{aligned}$$

The last four lines could have also been replaced by the following:

$$\begin{aligned}
PQ \parallel RS &\iff \cdots \\
&\iff \frac{\cos\frac{1}{2}(q+p)}{-\sin\frac{1}{2}(q+p)} = \frac{\cos\frac{1}{2}(s+r)}{-\sin\frac{1}{2}(s+r)} \\
&\iff \cos\tfrac{1}{2}(q+p)\sin\tfrac{1}{2}(s+r) = \cos\tfrac{1}{2}(s+r)\sin\tfrac{1}{2}(q+p) \\
&\iff \sin\tfrac{1}{2}(s+r)\cos\tfrac{1}{2}(q+p) - \cos\tfrac{1}{2}(s+r)\sin\tfrac{1}{2}(q+p) = 0 \\
&\iff \sin\tfrac{1}{2}((q+p)-(s+r)) = 0 \\
&\iff \tfrac{1}{2}(q+p-s-r) = \pi \qquad \text{for some } k \in \mathbb{Z} \\
&\iff r+s-p-q = 2n\pi \qquad \text{for some } n \in \mathbb{Z}.
\end{aligned}$$

We are almost there; we now only need to show $n = 1$ in the final line. We know that $0 \leqslant p < q < r < s < 2\pi$, so $r+s < 4\pi$ and $0 < p+q < r+s$, so that $0 < r+s-p-q < 4\pi$, which means that n must equal 1 if PQ and RS are parallel.

Thus PQ and RS are parallel if and only if $r+s-p-q = 2\pi$.

Marks

M1: Writing down expressions for the gradients of PQ and RS (do not need to state or show that denominators are non-zero, as told that lines are not vertical)

M1 dep: Factorising the numerators and denominators of the gradients as $a(\cdots)$ and $b(\cdots)$ (can be implied by later work, such as dividing or multiplying both sides of an equation by a or b)

M1: Applying identities from first part to rewrite at least one of the gradient expressions ...

A1: ... correctly

A1: Other gradient similarly factorised correctly

M1: Equating expressions for the gradients of PQ and RS (or equivalent)

M1: Cancelling common factors

A1: Deducing equivalence of $\tan\frac{1}{2}(p+q)$ and $\tan\frac{1}{2}(r+s)$ or same for cotangents

Proving that parallel implies $r+s-p-q = 2\pi$:

M1: Deducing $p+q = r+s+2k\pi$; need to have $2k\pi$ or equivalent in this expression for this method mark, or some other indication that all of the relevant possibilities are being considered; alternatively, arguing that $k \neq 0$ gains this method mark

M1: Correctly arguing that $0 < r+s-p-q < 4\pi$

A1 cso: Deducing required implication

Proving that $r+s-p-q = 2\pi$ implies parallel:

B1: Either ensuring that the argument is bi-directional ("if and only if") or for showing that if $r+s-p-q = 2\pi$, then PQ is parallel to RS

[Total for this part: 12 marks]

Question 4

Use the substitution $x = \dfrac{1}{t^2-1}$*, where* $t > 1$*, to show that, for* $x > 0$*,*

$$\int \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x}+\sqrt{x+1}\right) + c.$$

*[**Note:** You may use without proof the result* $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| +$ *constant.]*

Using the given substitution, we first use the chain rule to calculate

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(t^2-1)^{-2}\cdot 2t = -\frac{2t}{(t^2-1)^2}.$$

(We could alternatively have used the quotient rule to reach the same conclusion.)

We can now perform the requested substitution, simplifying the algebra as we go:

$$\begin{aligned}
\int \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{\frac{1}{t^2-1}\cdot\frac{t^2}{t^2-1}}}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{1}{\left(\frac{t}{t^2-1}\right)}\cdot\frac{-2t}{(t^2-1)^2}\,\mathrm{d}t \\
&= \int \frac{-2}{t^2-1}\,\mathrm{d}t \\
&= -2\times\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c \qquad \text{using the given result} \\
&= \ln\left|\frac{t+1}{t-1}\right| + c.
\end{aligned}$$

At this point, we wish to substitute t for x, so we rearrange the original substitution to get

$$t = \sqrt{\frac{1}{x}+1} = \sqrt{\frac{x+1}{x}}.$$

This now yields:

$$\ln\left|\frac{t+1}{t-1}\right| + c = \ln\left|\frac{\sqrt{\frac{x+1}{x}}+1}{\sqrt{\frac{x+1}{x}}-1}\right| + c.$$

We note immediately that we can drop the absolute value signs, since both the numerator and denominator of the fraction are positive (the denominator is positive as $t > 1$ or $\sqrt{(x+1)/x} > 1$). So we get, on multiplying the numerator and denominator of the fraction by $\sqrt{x}$ to clear the fractions,

$$
\begin{aligned}
\ln\left|\frac{t+1}{t-1}\right| + c &= \ln\left(\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) + c \\
&= \ln\left(\frac{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}\right) + c \\
&= \ln\left(\frac{\left(\sqrt{x+1}+\sqrt{x}\right)^2}{(x+1)-x}\right) + c \\
&= \ln\left(\sqrt{x+1}+\sqrt{x}\right)^2 + c \\
&= 2\ln\left(\sqrt{x+1}+\sqrt{x}\right) + c,
\end{aligned}
$$

which is what we were after.

Marks

M1: Using chain rule or quotient rule to calculate $\frac{\mathrm{d}x}{\mathrm{d}t}$

A1: Correct derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ *(need not be simplified)*

M1: Attempt to perform substitution correctly (requires both substituting for x *and replacing* $\mathrm{d}x$ *by* $\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ *or equivalent; do not need to have yet calculated* $\frac{\mathrm{d}x}{\mathrm{d}t}$*)*

M1 dep: Simplifying square root in integrand and substituting in value of $\frac{\mathrm{d}x}{\mathrm{d}t}$

A1: Simplifying integrand to $-2/(t^2-1)$

A1: Correct evaluation of integral in terms of t *(using integral given in question or otherwise)*

B1: Correct rearrangement for t *in terms of* x *(condone use of* $\pm$*)*

The remaining marks in this part of the question can only be awarded if the candidate has reached an integral of the form $k\ln(t-a)/(t+a)$ *or* $k\ln|(t-a)/(t+a)|$*, where* k *and* a *are constants.*

B1: Justifying removal of absolute value signs (at any point in argument), or dropping them from around $|blah^2|$*; the final A1 can be awarded even without this*

M1: Substituting $t=\sqrt{(x+1)/x}$ *into integral*

M1: Multiplying through by $\sqrt{x}$ *to simplify fraction*

M1: Rationalising denominator

A1 cso: Reaching given answer

Alternative approach for final six *marks of this part: take the given answer and substitute* $x=1/(t^2-1)$ *into it, showing that they end up with the integral they have worked out. Again, these can only be awarded if the candidate has already reached an integral of the aforementioned form. If the candidate is going down this route, they can only receive the B1 mark for rearranging for* t *in terms of* x *if they gain no marks from this alternative mark scheme.*

M1: Attempting to substitute $x=1/(t^2-1)$ *into the log expression*

A1: Simplifying the second root to $t/\sqrt{t^2-1}$

M1: Squaring the expression inside the logarithm …

A1: ... correctly

A1 cso: Simplifying the result to reach the integral in terms of t

B1: Justifying removal of absolute value signs from the logarithm (at any point in argument), or dropping them from around $|blah^2|$; the final A1 can be awarded even without this

[Total for this part: 12 marks]

The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360° about the x-axis. Show that the volume enclosed is $2\pi \ln \frac{5}{4}$.

To find the volume of revolution, we need to calculate the definite integral $\int_{1/8}^{9/16} \pi y^2 \,\mathrm{d}x$:

$$\begin{aligned}
&\int_{1/8}^{9/16} \pi y^2 \,\mathrm{d}x \\
&\quad = \pi \int_{1/8}^{9/16} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right)^2 \mathrm{d}x \\
&\quad = \pi \int_{1/8}^{9/16} \frac{1}{x} - 2\frac{1}{\sqrt{x(x+1)}} + \frac{1}{x+1} \,\mathrm{d}x \\
&\quad = \pi \Big[\ln x - 4\ln\big(\sqrt{x} + \sqrt{x+1}\big) + \ln(x+1)\Big]_{1/8}^{9/16} \qquad \text{using the above result} \\
&\quad = \pi\Big(\ln \tfrac{9}{16} - 4\ln\big(\sqrt{\tfrac{9}{16}} + \sqrt{\tfrac{25}{16}}\big) + \ln\tfrac{25}{16}\Big) - \pi\Big(\ln\tfrac{1}{8} - 4\ln\big(\sqrt{\tfrac{1}{8}} + \sqrt{\tfrac{9}{8}}\big) + \ln\tfrac{9}{8}\Big) \\
&\quad = \pi\Big(2\ln\tfrac{3}{4} - 4\ln(\tfrac{3}{4} + \tfrac{5}{4}) + 2\ln\tfrac{5}{4}\Big) - \pi\Big(2\ln\tfrac{1}{2\sqrt{2}} - 4\ln\big(\tfrac{1}{2\sqrt{2}} + \tfrac{3}{2\sqrt{2}}\big) + 2\ln\tfrac{3}{2\sqrt{2}}\Big) \\
&\quad = \pi\big((2\ln 3 - 2\ln 4) - 4\ln 2 + (2\ln 5 - 2\ln 4)\big) - \\
&\qquad\quad \pi\big(-2\ln(2\sqrt{2}) - 4\ln\sqrt{2} + (2\ln 3 - 2\ln(2\sqrt{2}))\big) \\
&\quad = \pi(2\ln 3 - 4\ln 2 - 4\ln 2 + 2\ln 5 - 4\ln 2) - \\
&\qquad\quad \pi(-3\ln 2 - 2\ln 2 + 2\ln 3 - 3\ln 2) \\
&\quad = \pi(-4\ln 2 + 2\ln 5) \\
&\quad = 2\pi(-2\ln 2 + \ln 5) \\
&\quad = 2\pi \ln\tfrac{5}{4}.
\end{aligned}$$

Marks

M1: Use of $\int \pi y^2 \,\mathrm{d}x$ (condone $\int 2\pi y^2 \,\mathrm{d}x$ for this method mark, this will just lose them the final accuracy mark)

A1: Correct expanded integrand

A1: Correct integration of all three terms

M1: Substitution of limits into integral (condone at most one sign error)

M1: Simplifying logs of square root expressions to a single log

M1: Simplifying logs of fractions and roots to give expressions in terms of $\ln 2$, $\ln 3$ *and* $\ln 5$ *(simplification of at least three terms correctly sufficient for this mark)*

A1: Reaching a correct unsimplified expression in terms of $\ln 2$, $\ln 3$ *and* $\ln 5$

A1 cso: Correct answer in given form, correctly deduced

[Total for this part: 8 marks]

Question 5

By considering the expansion of $(1+x)^n$ *where* n *is a positive integer, or otherwise, show that:*

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$;

We take the advice and begin by writing out the expansion of $(1+x)^n$:

$$(1+x)^n=\binom{n}{0}x^0+\binom{n}{1}x^1+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n, \qquad (*)$$

where we have pedantically written in x^0 and x^1 in the first two terms, as this may well help us to understand what we are looking at.

Now comparing this expansion to the expression we are interested in, we see that the only difference is the presence of the xs. If we substitute $x=1$, we will get exactly what we want:

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n},$$

as all powers of 1 are just 1.

Marks

B1: Correctly writing out the binomial expansion of $(1+x)^n$ *in terms of binomial coefficients and powers of* x *(allow powers of* 1 *in the terms, and presence or not of* x^0*)*

M1: Substituting $x=1$

A1 cso: Reaching required conclusion

[Total for this part: 3 marks]

(ii) $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n2^{n-1}$;

For the rest of the question, there are two very distinct approaches, one via calculus and one via properties of binomial coefficients.

Approach 1: Use calculus

This one looks a little more challenging, and we must observe carefully that there is no $\binom{n}{0}$ term. Comparing to the binomial expansion, we see that the term $\binom{n}{r}x^r$ has turned into $\binom{n}{r}.r$. Now, setting $x=1$ will again remove the x, but where are we to get the r from? Calculus gives us the answer: if we differentiate with respect to x, then x^r becomes

rx^{r-1}, and then setting $x = 1$ will complete the job. Now differentiating $(*)$ gives

$$n(1+x)^{n-1} = \binom{n}{1}.1x^0 + \binom{n}{2}.2x^1 + \binom{n}{3}.3x^2 + \cdots + \binom{n}{n}.nx^{n-1},$$

so by setting $x = 1$, we get the desired result.

Approach 2: Use properties of binomial coefficients

We know that

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

so we can manipulate this formula to pull out an r, using $r! = r.(r-1)!$ and similar expressions. We get

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{(n-r)!r!} \\ &= \frac{1}{r}.\frac{n!}{(n-r)!(r-1)!} \\ &= \frac{n}{r}.\frac{(n-1)!}{(n-r)!(r-1)!} \\ &= \frac{n}{r}\binom{n-1}{r-1}\end{aligned}$$

so that $r\binom{n}{r} = n\binom{n-1}{r-1}$. This is true as long as $r \geqslant 1$ and $n \geqslant 1$, so we get

$$\begin{aligned}\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} &= n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1} \\ &= n.2^{n-1}\end{aligned}$$

where we have used the result from part (i) with $n-1$ in place of n to do the last step.

Marks

In each of the remaining parts of this question, use whichever markscheme gives higher marks if both approaches attempted.

Approach 1:

M1: Attempting to differentiate either the binomial expansion of $(1+x)^n$ or the expression $(1+x)^n$ itself

A1: Correctly differentiating series term-by-term (including losing the constant term)

A1: Correctly differentiating $(1+x)^n$

M1: Substituting $x = 1$

A1 cso: Reaching desired result

Approach 2:

M1: Writing out $n(1+1)^{n-1}$ in terms of factorials …

A1 cao: ... correctly, i.e., including first and last terms correct
M1: Rearranging to get something that looks like $\sum r\binom{n}{r}$ (at the start and end)
M1: Attempting to justify middle terms, by showing that $n\binom{n-1}{r-1} = r\binom{n}{r}$
A1 cso: Fully correct argument
[Total for this part: 5 marks]

(iii) $\displaystyle \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}\left(2^{n+1} - 1\right);$

Approach 1: Use calculus

Spurred on by our previous success, we see that now the $\binom{n}{2}x^2$ term gives us $\binom{n}{2}.\frac{1}{3}$, so we think of integration instead. Integrating $(*)$ gives

$$\frac{1}{n+1}(1+x)^{n+1} = \binom{n}{0}.x^1 + \binom{n}{1}.\frac{1}{2}x^2 + \binom{n}{2}.\frac{1}{3}x^3 + \cdots + \binom{n}{n}.\frac{1}{n+1}x^{n+1} + c\,.$$

We do need to determine the constant of integration, so we put $x = 0$ to do this; this gives

$$\frac{1}{n+1}1^{n+1} = \binom{n}{0}.0 + \binom{n}{1}.\frac{1}{2}.0 + + \cdots + \binom{n}{n}.\frac{1}{n+1}.0 + c,$$

so $c = \frac{1}{n+1}$. Now substituting $x = 1$ gives

$$\frac{1}{n+1}(1+1)^{n+1} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + +\frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} + \frac{1}{n+1}.$$

Finally, subtracting $\frac{1}{n+1}$ from both sides gives us our required result.

(An alternative way to think about this is to integrate both sides from $x = 0$ to $x = 1$.)

Approach 2: Use properties of binomial coefficients

We can try rewriting our identity so that the $\frac{1}{r}$ stays with the $r - 1$ term; this gives us

$$\frac{1}{n}\binom{n}{r} = \frac{1}{r}\binom{n-1}{r-1}.$$

Unfortunately, though, our expressions involve $\binom{n}{r-1}$ terms rather than $\binom{n-1}{r-1}$ terms, but we can fix this by replacing n by $n + 1$ to get

$$\frac{1}{n+1}\binom{n+1}{r} = \frac{1}{r}\binom{n}{r-1}.$$

We substitute this in to get

$$\begin{aligned}
&\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \cdots + \frac{1}{n+1}\binom{n}{n} \\
&\qquad = \frac{1}{n+1}\binom{n+1}{1} + \frac{1}{n+1}\binom{n+1}{2} + \cdots + \frac{1}{n+1}\binom{n+1}{n+1} \\
&\qquad = \frac{1}{n+1}(2^{n+1} - 1),
\end{aligned}$$

where we have again used the result of part (i), this time with $n+1$ replacing n.

Marks

Approach 1:

M1: Attempting to integrate binomial expansion or $(1+x)^n$
A1: Correctly integrating binomial expansion (ignore limits and "$+c$" for this mark)
A1: Correctly integrating $(1+x)^n$ (again, ignoring limits or $+c$)
M1: Using limits of integration 0 to a (any a) or attempting to evaluate both sides at some specific value of x in order to determine c
A1: Correct determination of c or choice of $a=1$ and evaluating both sides correctly
A1 cso: Reaching stated result

Approach 2:

M1: Writing out $\frac{1}{n+1}\big((1+1)^{n+1}-1\big)$ in terms of factorials ...
A1: ... accurately: first and last terms and -1 all accounted for
M1: Rearranging to get $\sum \frac{1}{r+1}\binom{n}{r}$ or similar
M1: Attempting to justify middle terms ...
A1: ... accurately
A1 cso: Putting everything together correctly
[Total for this part: 6 marks]

(iv) $\displaystyle \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n(n+1)2^{n-2}.$

Approach 1: Use calculus

This looks similar to (ii), in that we have increasing multiples. So we try differentiating $(*)$ twice, giving us:

$$n(n-1)(1+x)^{n-2} = \binom{n}{1}.1.0 + \binom{n}{2}.2.1x^0 + \binom{n}{3}.3.2x^1 + \cdots + \binom{n}{n}.n(n-1)x^{n-2}.$$

Unfortunately, though, the coefficient of $\binom{n}{r}$ is $r(r-1)$ rather than the r^2 we actually want. But no matter: we can just add r and we will be done, as $r^2 = r(r-1)+r$, and we know from (ii) what terms like $\binom{n}{2}.2$ sum to give us. So we have, putting $x=1$ in our above expression:

$$n(n-1)(1+1)^{n-2} = \binom{n}{1}.1.0 + \binom{n}{2}.2.1 + \binom{n}{3}.3.2 + \cdots + \binom{n}{n}.n(n-1).$$

Now adding the result of (ii) gives

$$n(n-1)2^{n-2}+n2^{n-1}=\binom{n}{1}.(1.0+1)+\binom{n}{2}.(2.1+2)+\binom{n}{3}.(3.2+3)+\cdots+\binom{n}{n}.(n(n-1)+n)$$

so

$$(n(n-1)+2n)2^{n-2}=\binom{n}{1}+\binom{n}{2}.2^2+\binom{n}{3}.3^2+\cdots+\binom{n}{n}.n^2.$$

The left side simplifies to $n(n+1)2^{n-2}$, and thus we are done.

An alternative (calculus-based) method is as follows. The first derivative of $(*)$, as we have seen, is

$$n(1+x)^{n-1}=\binom{n}{1}.1x^0+\binom{n}{2}.2x^1+\binom{n}{3}.3x^2+\cdots+\binom{n}{n}.nx^{n-1}.$$

Now were we to differentiate again, we would end up with terms like $n(n-1)x^{n-2}$, rather than the desired n^2x^k (for some k). We can remedy this problem by multiplying the whole identity by x before we differentiate, so that we are differentiating

$$nx(1+x)^{n-1}=\binom{n}{1}.1x^1+\binom{n}{2}.2x^2+\binom{n}{3}.3x^3+\cdots+\binom{n}{n}.nx^n.$$

Differentiating this now gives (using the product rule for the left hand side):

$$n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2}=\binom{n}{1}.1^2x^0+\binom{n}{2}.2^2x^1+\binom{n}{3}.3^2x^2+\cdots+\binom{n}{n}.n^2x^{n-1}.$$

Substituting $x=1$ into this gives our desired conclusion (after a small amount of algebra on the left hand side).

Approach 2: Use properties of binomial coefficients

As this looks similar to the result of part (ii), we can start with what we worked out there, namely $r\binom{n}{r}=n\binom{n-1}{r-1}$, giving us

$$\binom{n}{1}+2^2\binom{n}{2}+\cdots+n^2\binom{n}{n}=n\binom{n-1}{0}+n.2\binom{n-1}{1}+n.3\binom{n-1}{2}+\cdots+n.n\binom{n-1}{n-1}$$

Taking out the factor of n leaves us having to work out

$$\binom{n-1}{0}+2\binom{n-1}{1}+3\binom{n-1}{2}+\cdots+n\binom{n-1}{n-1}$$

This looks very similar to the problem of part (ii) with n replaced by $n-1$, but now the multiplier of $\binom{n}{r}$ is $r+1$ rather than r, and there is also an $\binom{n}{0}$ term. We can get over the first problem by splitting up $(r+1)\binom{n}{r}$ as $r\binom{n}{r}+\binom{n}{r}$, so this expression becomes

$$\begin{aligned}&\binom{n-1}{1}+2\binom{n-1}{2}+\cdots+(n-1)\binom{n-1}{n-1}+\\ \binom{n-1}{0}+&\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-1}\end{aligned}$$

The first line is just part (ii) with n replaced by $n-1$, so that it sums to $(n-1).2^{n-2}$, and the second line is just $2^{n-1} = 2.2^{n-2}$ by part (i). So the answer to the original question (remembering the factor of n we took out earlier) is

$$n\big((n-1).2^{n-2} + 2.2^{n-2}\big) = n(n-1+2).2^{n-2} = n(n+1).2^{n-2}.$$

Marks

Approach 1:

M1: Differentiating $(*)$ twice
M1: Recognising that $r^2 = r(r-1) + r$ and making use of this fact
M1: Substituting $x = 1$ to get an expression for $n(n-1)2^{n-2}$ in terms of binomial coefficients
M1: Adding result of (ii) and collecting terms appropriately
M1: Simplifying the left hand side correctly
A1 cso: Reaching the stated result

Alternative calculus method (likely to be incredibly rare; please discuss if you have questions about a script which uses this approach):

M1: Multiplying derivative of $(*)$ by x
M1: Differentiating this result ...
A1: ... correctly
M1: Substituting $x = 1$ into result
M1: Simplifying the left hand side correctly
A1 cso: Reaching the stated result

Approach 2:

M1: Applying technique from part (ii) once
M1: Splitting up $r+1$ into two parts
M1: Splitting sum as $\sum r\binom{n-1}{r} + \sum \binom{n-1}{r}$
M1: Summing $\sum r\binom{n-1}{r}$ using part (ii) result or otherwise
A1 ft: Determining sum of $\binom{(}{r} + 1)n - 1r$
A1 cso: Deriving the stated result
[Total for this part: 6 marks]

Question 6

Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0. \qquad (*)$$

If $y = \mathrm{e}^x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^x$. Substituting these into the left hand side of $(*)$ gives

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = (x-1)\mathrm{e}^x - x\mathrm{e}^x + \mathrm{e}^x = 0,$$

so $y = \mathrm{e}^x$ satisfies $(*)$.

Marks
B1: Correctly differentiating y twice
A1 cso: Substituting and deducing the required result
[Total for this part: 2 marks]

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \qquad (**)$$

We have $y = u\mathrm{e}^x$, so we apply the product rule to get:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x \\
&= \left(\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{e}^x + \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x\right) + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x\right) \\
&= \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{e}^x + 2\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x \\
&= \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x.
\end{aligned}$$

(If you know Leibniz's Theorem, then you could write down $\mathrm{d}^2u/\mathrm{d}x^2$ directly.)

We now substitute these into $(*)$ to get

$$(x-1)\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x - x\left(\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x + u\mathrm{e}^x = 0.$$

Dividing by $\mathrm{e}^x \neq 0$ and collecting the derivatives of u then gives

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (2x-2-x)\frac{\mathrm{d}u}{\mathrm{d}x} + (x-1-x+1)u = 0,$$

which gives $(**)$ on simplifying the brackets.

Marks

M1: Differentiating $u\mathrm{e}^x$ using the product rule

A1: Correct $\mathrm{d}y/\mathrm{d}x$

M1: Attempting to differentiate $\mathrm{d}y/\mathrm{d}x$ using product rule for $(\mathrm{d}u/\mathrm{d}x)\mathrm{e}^x$ term (implied if $\mathrm{d}^2y/\mathrm{d}x^2$ written down correctly with no working as they may have used Leibniz's Theorem)

A1: Correct derivative of $(\mathrm{d}u/\mathrm{d}x)\mathrm{e}^x$ term

A1 cao: Correct $\mathrm{d}^2y/\mathrm{d}x^2$ (whether simplified or not)

M1 dep: Substituting derivatives into $(*)$ and simplifying (dependent on both previous method marks)

A1 cso: Reaching required form

[Total for this part: 7 marks]

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

As instructed, we set $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$, so that $\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$, which gives us

$$(x-1)\frac{\mathrm{d}v}{\mathrm{d}x} + (x-2)v = 0.$$

This is a standard separable first-order linear differential equation, so we separate the variables to get

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x-2}{x-1}$$

and then integrate with respect to x to get

$$\int \frac{1}{v}\,\mathrm{d}v = \int -\frac{x-2}{x-1}\,\mathrm{d}x.$$

Performing the integrations now gives us

$$\begin{aligned}\ln|v| &= \int -\Bigl(1 - \frac{1}{x-1}\Bigr)\,\mathrm{d}x \\ &= -x + \ln|x-1| + c,\end{aligned}$$

which we exponentiate to get

$$|v| = |k|\mathrm{e}^{-x}|x-1|,$$

where k is some constant, so we finally arrive at

$$v = k\mathrm{e}^{-x}(x-1).$$

We now recall that $v = \mathrm{d}u/\mathrm{d}x$, so we need to integrate this last expression once more to find u. We use integration by parts to do this, integrating the e^{-x} part and differentiating $(x-1)$, to give us

$$\begin{aligned} u &= \int k\mathrm{e}^{-x}(x-1)\,\mathrm{d}x \\ &= k(-\mathrm{e}^{-x})(x-1) - \int k(-\mathrm{e}^{-x}).1\,\mathrm{d}x \\ &= k(-\mathrm{e}^{-x})(x-1) - k\mathrm{e}^{-x} + c \\ &= -kx\mathrm{e}^{-x} + c, \end{aligned}$$

which is the solution to ($**$).

Now recalling that $y = u\mathrm{e}^x$ gives us $y = -kx + c\,\mathrm{e}^x$ as our general solution to ($*$). In particular, letting $k = -A$ and $c = B$, where A and B are any constants, shows that $y = Ax + B\mathrm{e}^x$ satisfies ($*$), as required.

Marks

B1: Stating (or using) $\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

B1: Correct first-order ODE for v

M1: Separating variables

M1: Reasonable attempt to integrate $(x-2)/(x-1)$ *either by dividing through or by substituting* $u = x-1$ *or other appropriate method*

A1 cso: Reaching a correct expression for $\ln|v|$*; condone absence of absolute value signs and missing "*$+c$*" for this mark*

M1: Exponentiating to find an expression for $|v|$ *or* v

A1 cso: Correct expression for $|v|$ *or* v*, which must include an arbitrary constant*

Alternative using integrating factors for preceding 5 marks:

M1: Reasonable attempt at using an integrating factor, reaching $\exp\int(x-2)/(x-1)\,\mathrm{d}x$ *or better*

M1 dep: Reasonable attempt to integrate $(x-2)/(x-1)$ *either by dividing through or by substituting* $u = x-1$ *or other appropriate method*

A1 cso: Reaching a correct expression for $\int(x-2)/(x-1)\,\mathrm{d}x$ *(no arbitrary constant necessary here, of course; condone absence of absolute value signs)*

M1: Reaching $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^x}{x-1}v\right) = 0$ *or better*

A1 cso: Correct expression for $|v|$ *or* v*, which must include an arbitrary constant*

M1: Integrating $v = \mathrm{d}u/\mathrm{d}x$ *by parts (need to have absorbed any modulus sign into the arbitrary constant to be able to do this!)*

A1 cso: Correct expression for u involving two arbitrary constants

M1: Substituting u into $y = u\mathrm{e}^x$ to get an expression for y (can award even if missing one of the two arbitrary constants)

A1 cso: Setting arbitrary constants to appropriate functions of A and B to reach desired conclusion (may be automatic by earlier choice of arbitrary constants, but has to be done without fudging!)

[Total for this part: 11 marks]

Question 7

Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$, and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta},$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

The condition $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ with $\alpha + \beta < 1$ and α and β both positive constants means that C lies strictly inside the triangle OAB. Can you see why?

We start by sketching the setup so that we have something visual to help us with our thinking.

The line OA has points with position vectors given by $\mathbf{r}_1 = \lambda\mathbf{a}$, and the line BC has points with position vectors given by

$$\mathbf{r}_2 = \overrightarrow{OB} + \mu\overrightarrow{BC} = \mathbf{b} + \mu(\mathbf{c} - \mathbf{b}) = (1-\mu)\mathbf{b} + \mu\mathbf{c}.$$

The point P is where these two lines meet, so we must have

$$\begin{aligned}\mathbf{p} = \lambda\mathbf{a} &= (1-\mu)\mathbf{b} + \mu\mathbf{c} \\ &= (1-\mu)\mathbf{b} + \mu(\alpha\mathbf{a} + \beta\mathbf{b}) \\ &= (1-\mu+\beta\mu)\mathbf{b} + \alpha\mu\mathbf{a}.\end{aligned}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, we must have $1 - \mu + \beta\mu = 0$ and $\alpha\mu = \lambda$. The first equation gives $(1-\beta)\mu = 1$, so $\mu = 1/(1-\beta)$. This gives $\lambda = \alpha/(1-\beta)$, so that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}.$$

Now swapping the roles of $\mathbf{a}$ and $\mathbf{b}$ (and hence also of α and β) will give us the position vector of Q:

$$\mathbf{q} = \frac{\beta\mathbf{b}}{1-\alpha}.$$

Marks

B1: Vector equation of line OA (may be implied by later working; must be clear that it is the equation of the line *OA and not the* vector *$\overrightarrow{OA}$)*

B1: Vector equation of line BC (again may be implied, and again must clearly be the equation of the line and not a vector)

SC: If both vector equations are correct, but both are claimed to be vectors rather than equations of lines, award B0 B1

M1: Substituting for $\mathbf{c}$ *and rearranging to collect terms in* $\mathbf{a}$ *and* $\mathbf{b}$

M1: Equating equations in an attempt to find $\mathbf{p}$ *(can be implied by next method mark)*

M1 dep: Deducing that coefficients of ($\mathbf{a}$ *and)* $\mathbf{b}$ *must be equal, and giving the resulting equation(s) (again, can be implied)*

M1 dep: A reasonable attempt to solve these simultaneous equations or just the equation for coefficients of $\mathbf{b}$

A1 cso: Deducing stated formula for $\mathbf{p}$*; need to be clear that point lies on both lines (either by earlier working or by being explicit at this point) to gain this final answer mark; can still award if earlier SC awarded B0 B1*

B1 cao: Correct formula for $\mathbf{q}$ *(no credit for any working, but mark cannot be awarded if result deduced from incorrect working; if, however, working given improves on earlier attempt to deduce* $\mathbf{p}$*, please discuss with PE)*

SC: If no marks awarded for this part, but candidate has drawn a correct sketch including at least P and Q, and with C strictly inside OAB, then award B1

SC: If the candidate assumes that this question is two dimensional, they cannot gain the final A1 mark; they may, however, be awarded the rest of the method marks and the first two B marks, suitably interpreted.

[Total for this part: 8 marks]

Show further that the point R with position vector $\mathbf{r}$ *given by*

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}$$

lies on the lines OC and AB.

We could approach this question in two ways, either by finding the point of intersection of OC and AB or by showing that the given point lies on both given lines. We give both approaches.

Approach 1: Finding the point of intersection

We require $\mathbf{r}$ to lie on OC, so $\mathbf{r} = \lambda\mathbf{c}$, and $\mathbf{r}$ to lie on AB, so $\mathbf{r} = (1-\mu)\mathbf{a} + \mu\mathbf{b}$, as before. Substituting for $\mathbf{c}$ and equating coefficients gives

$$\alpha\lambda\mathbf{a} + \beta\lambda\mathbf{b} = (1-\mu)\mathbf{a} + \mu\mathbf{b},$$

so that

$$\begin{aligned}\alpha\lambda &= 1 - \mu \\ \beta\lambda &= \mu.\end{aligned}$$

Adding the two equations gives $(\alpha + \beta)\lambda = 1$, so $\lambda = 1/(\alpha + \beta)$ and hence

$$\mathbf{r} = \frac{\mathbf{c}}{\alpha + \beta} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}.$$

Approach 2: Showing that the given point lies on both lines

The equation of line OC is $\mathbf{r}_1 = \lambda\mathbf{c}$, and

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} = \frac{1}{\alpha + \beta}\mathbf{c}$$

is of the required form, so R lies on OC.

The equation of the line AB can be written as $\mathbf{r}_2 - \mathbf{a} = \mu\overrightarrow{AB}$, so we want $\mathbf{r} - \mathbf{a} = \mu(\mathbf{b} - \mathbf{a})$. Now we have

$$\begin{aligned}\mathbf{r} - \mathbf{a} &= \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} - \frac{\alpha + \beta}{\alpha + \beta}\mathbf{a} \\ &= \frac{-\beta\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} \\ &= \frac{\beta}{\alpha + \beta}(\mathbf{b} - \mathbf{a}),\end{aligned}$$

which is of the form $\mu(\mathbf{b} - \mathbf{a})$, so R lies on both lines.

Marks

Approach 1:

M1: Stating that we need to solve $\lambda\mathbf{c} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$ *or equivalent*
M1: Expanding $\mathbf{c}$ *and rearranging to equate coefficients of* $\mathbf{a}$ *and* $\mathbf{b}$
M1: Solving resulting simultaneous equations
A1 cso: Deducing that $\mathbf{r}$ *is as stated*

Approach 2:

B1: Showing $\mathbf{r}$ *lies on* OC
M1: Stating that we need to solve $\mathbf{r} = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$ *or equivalent*
M1: Rearranging expression for $\mathbf{r}$ *to find a suitable value of* μ *or equivalent*
A1 cso: Showing that $\mathbf{r}$ *lies on* AB
[Total for this part: 4 marks]

The lines OB *and* PR *intersect at the point* S*. Prove that* $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

S lies on both OB and PR, so we need to find its position vector, $\mathbf{s}$. Once again, we require $\mathbf{s} = \lambda\mathbf{b} = (1-\mu)\mathbf{p} + \mu\mathbf{r}$, so we substitute for $\mathbf{p}$ and $\mathbf{r}$ and compare coefficients:

$$\begin{aligned}
\mathbf{s} = \lambda\mathbf{b} &= (1-\mu)\mathbf{p} + \mu\mathbf{r} \\
&= (1-\mu)\frac{\alpha\mathbf{a}}{1-\beta} + \mu\frac{\alpha\mathbf{a}+\beta\mathbf{b}}{\alpha+\beta} \\
&= \frac{(1-\mu)\alpha(\alpha+\beta)\mathbf{a} + \mu(1-\beta)(\alpha\mathbf{a}+\beta\mathbf{b})}{(1-\beta)(\alpha+\beta)} \\
&= \frac{((1-\mu)\alpha(\alpha+\beta) + \mu\alpha(1-\beta))\mathbf{a} + \mu(1-\beta)\beta\mathbf{b}}{(1-\beta)(\alpha+\beta)} \\
&= \frac{\alpha(\alpha+\beta-\mu\alpha-2\mu\beta+\mu)\mathbf{a} + \mu(1-\beta)\beta\mathbf{b}}{(1-\beta)(\alpha+\beta)}
\end{aligned}$$

Since the coefficient of $\mathbf{a}$ in this expression must be zero, we deduce that

$$\mu = \frac{\alpha+\beta}{\alpha+2\beta-1},$$

so that

$$\begin{aligned}
\mathbf{s} &= \frac{\mu(1-\beta)\beta}{(1-\beta)(\alpha+\beta)}\mathbf{b} \\
&= \frac{\alpha+\beta}{\alpha+2\beta-1}\frac{\beta}{(\alpha+\beta)}\mathbf{b} \\
&= \frac{\beta}{\alpha+2\beta-1}\mathbf{b}
\end{aligned}$$

Now, since $\overrightarrow{OQ}$ and $\overrightarrow{BQ}$ are both multiples of the vector q, we can compare the lengths OQ and BQ in terms of their multiples of $\mathbf{q}$. This might come out to be negative, depending on the relative directions, but at the end, we can just consider the magnitudes.

We thus have, since $\mathbf{q} = \dfrac{\beta\mathbf{b}}{1-\alpha}$,

$$\begin{aligned}
\frac{OQ}{BQ} &= \Big(\frac{\beta}{1-\alpha}\Big) \Big/ \Big(\frac{\beta}{1-\alpha} - 1\Big) \\
&= \Big(\frac{\beta}{1-\alpha}\Big) \Big/ \Big(\frac{\beta-1+\alpha}{1-\alpha}\Big) \\
&= \frac{\beta}{\beta-1+\alpha},
\end{aligned}$$

while

$$\begin{aligned}
\frac{OS}{BS} &= \Big(\frac{\beta}{\alpha+2\beta-1}\Big) \Big/ \Big(\frac{\beta}{\alpha+2\beta-1} - 1\Big) \\
&= \Big(\frac{\beta}{\alpha+2\beta-1}\Big) \Big/ \Big(\frac{-\alpha-\beta+1}{\alpha+2\beta-1}\Big) \\
&= \frac{\beta}{-\alpha-\beta+1}.
\end{aligned}$$

Thus these two ratios of lengths are equal, as the magnitude of both of these is

$$\frac{\beta}{1-(\alpha+\beta)}.$$

Marks

NB: There are multiple ways to find the position vector of S, and quite possibly to do the rest of this part as well. If the attempt approximately parallels this mark scheme, then award marks correspondingly; if it is very different, please discuss with the PE

M1: Attempt to find position vector of S by equating equations of lines

M1: Finding position vector of S by equating equations of lines and substituting for $\mathbf{p}$ *and* $\mathbf{r}$

M1: Solving to determine μ

A1 cao: Determining $\mathbf{s}$

B1: Formula for OQ/BQ

B1 ft: Formula for OS/BS

M1: Simplifying the two ratios so that they are either equal or $OS/BS = -OQ/BQ$

A1 cso: Deducing that $OQ/BQ = OS/BS$ (must explain that magnitudes are what matters if $OS/BS = -OQ/BQ$); must have determined OS/BS from correct working for this final accuracy mark

[Total for this part: 8 marks]

Question 8

> *(i) Suppose that a, b and c are integers that satisfy the equation*
>
> $$a^3 + 3b^3 = 9c^3.$$
>
> *Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.*

We have $a^3 = 9c^3 - 3b^3 = 3(3c^3 - b^3)$, so a^3 is a multiple of 3. But as 3 is prime, a itself must be divisible by 3. (Why is this? If 3 divides a product rs, then 3 must divide either r or s, as 3 is prime. Therefore since $a^3 = a.a.a$, and 3 divides a^3, it follows that 3 must divide one of the factors, that is, 3 must divide a.)

Now we can write $a = 3d$, where d is an integer. Therefore we have

$$(3d)^3 + 3b^3 = 9c^3,$$

which, on dividing by 3, gives

$$9d^3 + b^3 = 3c^3.$$

By the same argument, as $b^3 = 3(c^3 - 3d^3)$, it follows that b^3, and hence also b, is divisible by 3.

We repeat the same trick, writing $b = 3e$, where e is an integer, so that

$$9d^3 + (3e)^3 = 3c^3.$$

We again divide by 3 to get

$$3d^3 + 9e^3 = c^3,$$

so that c^3, and hence also c, is divisible by 3.

We then write $c = 3f$, where f is an integer, giving

$$3d^3 + 9e^3 = (3f)^3.$$

Finally, we divide this equation by 3 to get

$$d^3 + 3e^3 = 9f^3.$$

Note that this is the same equation that we started with, so if a, b, c are integers which satisfy the equation, then so are $d = a/3$, $e = b/3$ and $f = c/3$. We can repeat this process indefinitely, so that $a/3^n$, $b/3^n$ and $c/3^n$ are also integers which satisfy the equation. But if $a/3^n$ is an integer for all $n \geqslant 0$, we must have $a = 0$, and similarly for b and c.

Therefore the only integer solution is $a = b = c = 0$.

[In fact, we can say even more. If a, b and c are all *rational*, say $a = d/r$, $b = e/s$, $c = f/t$ (where d, e, f are integers and r, s, t are non-zero integers), then we have

$$\left(\frac{d}{r}\right)^3 + 3\left(\frac{e}{s}\right)^3 = 9\left(\frac{f}{t}\right)^3.$$

Now multiplying both sides by $(rst)^3$ gives

$$(dst)^3 + 3(ert)^3 = 9(frs)^3,$$

with dst, ert and frs all integers, and so they must all be zero, and hence $d = e = f = 0$. Therefore, the only *rational* solution is also $a = b = c = 0$.]

Marks

M1: Arguing that a^3 is divisible by 3 (writing $a^3 = 3(3c^3 - b^3)$ sufficient)

A1 cso: Arguing via property of prime numbers that a must be divisible by 3 (be lenient with this mark; any approximately correct argument should get credit)

B1: Writing $a = 3d$

M1: Substituting for a in equation and expanding $(3d)^3$

M1: Dividing equation by 3

A1 cso: Arguing that b must be divisible by 3 (again allowing for vaguely correct arguments as before, and condone if same argument as for $3|a$)

M1: Writing $b = 3e$ and substituting into equation

A1 cso: Arguing that c must be divisible by 3, and condone if same argument as for $3|a$

M1: Writing $c = 3f$, substituting into equation and dividing by 3

M1: Observing that resulting equation for d, e, f is identical to original equation

A1 cao: A reasonable attempt at an infinite descent argument

[Total for this part: 11 marks]

(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

We consider the final digit of fourth powers:

a	a^4	$2a^4$
0	0	0
1	1	2
2	6	2
3	1	2
4	6	2
5	5	0
6	6	2
7	1	2
8	6	2
9	1	2

So the last digits of fourth powers are all either 0, 5, 1 or 6, and of twice fourth powers are all either 0 or 2.

Also, $5r^4$ is a multiple of 5, so it must end in a 0 or a 5.

Therefore if $2q^4$ ends in 0 (that is, when q is a multiple of 5), the possibilities for the final digit of $p^4 + 2q^4$ are

$$(0 \text{ or } 1 \text{ or } 5 \text{ or } 6) + 0 = 0 \text{ or } 1 \text{ or } 5 \text{ or } 6,$$

so it can equal $5r^4$ (which ends in 0 or 5) only if p^4 ends in 0 or 5, which is exactly when p is a multiple of 5.

Similarly, if $2q^4$ ends in 2 (so q is not a multiple of 5), the possibilities for the final digit of $p^4 + 2q^4$ are

$$(0 \text{ or } 1 \text{ or } 5 \text{ or } 6) + 2 = 2 \text{ or } 3 \text{ or } 7 \text{ or } 8,$$

so it can not be equal to $5r^4$ (which ends in 0 or 5).

Therefore, if $p^4 + 2q^4 = 5r^4$, we must have p and q both being multiples of 5.

Now as in part (i), we write $p = 5a$ and $q = 5b$, where a and b are both integers, to get

$$(5a)^4 + 2(5b)^4 = 5r^4.$$

Dividing both sides by 5 gives

$$5^3a^4 + 2.5^3b^4 = r^4,$$

where we are using dot to mean multiplication, so as before, r^4 must be a multiple of 5 as the left hand side is $5(5^2a^4 + 2.5^2b^4)$. Thus, since 5 is prime, r itself must be divisible by 5. Then writing $r = 5c$ gives

$$5^3a^4 + 2.5^3b^4 = (5c)^4,$$

which yields

$$a^4 + 2b^4 = 5c^4$$

on dividing by 5^3.

So once again, if p, q, r give an integer solution to the equation, so do $a = p/5$, $b = q/5$ and $c = r/5$. Repeating this, so are $p/5^n$, $q/5^n$, $r/5^n$, and as before, this shows that the only integer solution is $p = q = r = 0$.

[Again, the same argument as before shows that this is also the only rational solution.]

Marks

M1: Listing final digits of fourth powers and twice fourth powers (also works if consider just squares)

A1: All correct

M1: Considering possible last digits of $p^4 + 2q^4$ in the case $5 \nmid p$ or in the case $5 \nmid q$ (have to consider the possibility $5 \nmid p$ and $5 \mid q$ or vice versa to get this method mark), or for something else reasonable

A1 cso: Concluding that p and q must both be divisible by 5

[If candidates use modular arithmetic or other alternative methods, please discuss marking with PE if they are not perfect.]

M1: Writing $p = 5a$ and $q = 5b$ and substituting into equation
M1: Expanding and dividing equation by 5
A1: Deducing that r is divisible by 5
M1: Writing $r = 5c$, substituting and dividing equation by 5^3 to reach original equation
A1 cso: Arguing that $p = q = r = 0$ by infinite descent (again, allow a vague argument)
[Total for this part: 9 marks]

This is an example of the use of Fermat's Method of Descent, which he used to prove one special case of his famous Last Theorem: he showed that $x^4 + y^4 = z^4$ has no positive integer solutions. In fact, he proved an even stronger result, namely that $x^4 + y^4 = z^2$ has no positive integer solutions.

Another approach to solving the first step of part (ii) of this problem is to use modular arithmetic, *where we only consider remainders when dividing by a certain fixed number. In this case, we would consider arithmetic modulo 5, so the only numbers to consider are 0, 1, 2, 3 and 4, and we want to solve $p^4 + 2q^4 \equiv 0 \pmod 5$, where $\equiv$ means "leaves the same remainder". Now a quick calculation shows that $p^4 \equiv 1$ unless $p \equiv 0$, while $2q^4 \equiv 2$ unless $q \equiv 0$, so that*

$$p^4 + 2q^4 \equiv (0 \text{ or } 1) + (0 \text{ or } 2) \equiv 0, 1, 2 \text{ or } 3 \pmod 5$$

with $p^4 + 2q^4 \equiv 0$ if and only if $p \equiv q \equiv 0$.

Incidentally, Fermat has another theorem relevant to this problem, which turns out to be relatively easy to prove (Fermat himself claimed to have done so), and is known as Fermat's Little Theorem. *This states that, if p is prime, then $a^{p-1} \equiv 1 \pmod p$ unless $a \equiv 0 \pmod p$. In our case, $p = 5$ gives $a^4 \equiv 1 \pmod 5$ unless $a \equiv 0 \pmod 5$, as we wanted.*

Question 9

The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane. Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

We start by redrawing the sketch, labelling the corners and indicating the centre of mass as G, as well as showing various useful lengths.

It is now clear that the distance of G from the wall is $a\cos\alpha$ (horizontal distance from wall to midpoint of AB) plus $b\sin\alpha$ (horizontal distance from midpoint of AB to G), so a total of $a\cos\alpha + b\sin\alpha$.

Also, in case it is useful later, we note that the vertical distance above the horizontal plane is, by a similar argument from the same sketch, $a\sin\alpha + b\cos\alpha$.

> ***Marks***
> *M1: Dividing the distance of G from wall into two computable parts*
> *A1: Calculating either $a\cos\alpha$ or $b\sin\alpha$ with sufficient justification*
> *A1 cso: Determining given answer*
> *[Total for this part: 3 marks]*

The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

$$a\cos(2\lambda + \alpha) = b\sin\alpha,$$

where $\tan\lambda = \mu$.

There are two approaches to this. One is to indicate the reaction and friction forces separately, while the other is to use the Three Forces Theorem. We show both of these.

Approach 1: All forces separately

We start by sketching the lamina again, this time showing the forces on the lamina, separating the normal reactions from the frictional forces.

We now resolve and take moments:

$$\begin{array}{ll} \mathscr{R}(\uparrow) & F_1 + R_2 - W = 0 \\ \mathscr{R}(\rightarrow) & R_1 - F_2 = 0 \\ \mathscr{M}(\widehat{A}) & W(a\cos\alpha + b\sin\alpha) - R_2.2a\cos\alpha + F_2.2a\sin\alpha = 0 \end{array}$$

Since friction is limiting at both points of contact, we have $F_1 = \mu R_1$ and $F_2 = \mu R_2$. Substituting these gives:

$$\begin{array}{lll} \mathscr{R}(\uparrow) & \mu R_1 + R_2 - W = 0 & (1) \\ \mathscr{R}(\rightarrow) & R_1 - \mu R_2 = 0 & (2) \\ \mathscr{M}(\widehat{A}) & W(a\cos\alpha + b\sin\alpha) - 2aR_2\cos\alpha + 2a\mu R_2\sin\alpha = 0 & (3) \end{array}$$

Equation (2) gives $R_1 = \mu R_2$, so we can substitute this into (1) to get $W = (1 + \mu^2)R_2$. Substituting this into (3) now leads to

$$(1 + \mu^2)R_2(a\cos\alpha + b\sin\alpha) = 2aR_2(\cos\alpha - \mu\sin\alpha).$$

We can clearly divide both sides by R_2, and we are given that $\tan\lambda = \mu$, so we substitute this in as well, to get

$$(1 + \tan^2\lambda)(a\cos\alpha + b\sin\alpha) = 2a(\cos\alpha - \tan\lambda\sin\alpha).$$

We spot $1 + \tan^2\lambda = \sec^2\lambda$, and so multiply the whole equation through by $\cos^2\lambda$, as the form we are looking for does not involve $\sec\lambda$:

$$a\cos\alpha + b\sin\alpha = 2a(\cos^2\lambda\cos\alpha - \sin\lambda\cos\lambda\sin\alpha).$$

Since the form we are going for is $b\sin\alpha = a\cos(2\lambda + \alpha)$, we make use of double angle formulæ, after rearranging:

$$\begin{aligned} b\sin\alpha &= a(2\cos^2\lambda\cos\alpha - 2\sin\lambda\cos\lambda\sin\alpha) - a\cos\alpha \\ &= a((2\cos^2\lambda - 1)\cos\alpha - 2\sin\lambda\cos\lambda\sin\alpha) \\ &= a(\cos 2\lambda\cos\alpha - \sin 2\lambda\sin\alpha) \\ &= a\cos(2\lambda + \alpha), \end{aligned}$$

and we are done with this part.

Marks

M1: Indicating all forces on diagram (condone at most one missing/extra force or one other error)
M1: Resolving in two directions …
A1: … correctly
M1: Taking moments around a suitable point …
A1: … correctly
B1: Using $F = \mu R$ correctly in context
M1: Solving resolved equations to get $W = (1 + \mu^2)R_2$ or similar
M1: Substituting this into moments equation to get a force-free equation
M1: Substituting $\tan\lambda = \mu$ to get an equation relating λ and α

M1: Using $1 + \tan^2\lambda = \sec^2\lambda$ *and multiplying through by* $\cos^2\lambda$ *to clear denominators or equivalent*
M1: Rearranging to get $b\sin\alpha$ *on one side and collecting* a *terms on the other*
M1: Applying double angle formulæ to get $\cos 2\lambda$ *and* $\sin 2\lambda$
M1: Applying compound angle formula to get desired result
A1 cso: Correct solution reaching stated result
[Total for this part: 14 marks]

Approach 2: Three Forces Theorem

The 'Three Forces Theorem' states that if three (non-zero) forces act on a large body in equilibrium, and they are not all parallel, then they must pass through a single point. (Why is this true? Let's say two of the forces pass through point X. Taking moments about X, the total moment must be zero, so the moment of the third force about X must be zero. Therefore, the force itself is either zero or it passes through X. Since the forces are non-zero, the third force must pass through X.)

In our case, we have a normal reaction and a friction force at each point of contact. We can combine these into a single reaction force as shown in the sketch. Here we have written N for the normal force, F for the friction and R for the resultant, which is at an angle of θ to the normal.

We see from this sketch that $\tan\theta = F/N = \mu N/N = \mu$. In our case, since $\tan\lambda = \mu$ we must have $\theta = \lambda$.

We can now redraw our original diagram with the three (combined) forces shown:

We can now use the Three Forces Theorem is as follows. Looking at the diagram, we know that the distance $OB = 2a\cos\alpha = OP + PB$. Now we know $OP = a\cos\alpha + b\sin\alpha$, so we need only calculate PB.

But $PB = PX\tan\lambda$ (using the triangle PBX), and

$$\begin{aligned} PX &= OA + \text{height of } X \text{ above } A \\ &= 2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda. \end{aligned}$$

Putting these together gives

$$\begin{aligned} OB = 2a\cos\alpha &= OP + PB \\ &= a\cos\alpha + b\sin\alpha + (2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda)\tan\lambda \\ &= a\cos\alpha(1+\tan^2\lambda) + b\sin\alpha(1+\tan^2\lambda) + 2a\sin\alpha\tan\lambda \\ &= a\cos\alpha\sec^2\lambda + b\sin\alpha\sec^2\lambda + 2a\sin\alpha\tan\lambda. \end{aligned}$$

We can now rearrange to get

$$b\sin\alpha\sec^2\lambda = 2a\cos\alpha - a\cos\alpha\sec^2\lambda - 2a\sin\alpha\tan\lambda.$$

Since we want an expression for $b\sin\alpha$, we now multiply by $\cos^2\lambda$ to get

$$\begin{aligned}
b\sin\alpha &= 2a\cos\alpha\cos^2\lambda - a\cos\alpha - 2a\sin\alpha\sin\lambda\cos\lambda \\
&= a((2\cos^2\lambda - 1)\cos\alpha - (2\sin\lambda\cos\lambda)\sin\alpha) \\
&= a(\cos 2\lambda\cos\alpha - \sin 2\lambda\sin\alpha) \\
&= a\cos(2\lambda + \alpha).
\end{aligned}$$

An alternative argument using the Three Forces Theorem proceeds by considering the distance XP. Using the left half of the diagram, we have

$$\begin{aligned}
XP &= OA + OP\tan\lambda \\
&= 2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda.
\end{aligned}$$

From the right half of the diagram, we also have $XP = PB/\tan\lambda$, and $PB = 2a\cos\alpha - OP = a\cos\alpha - b\sin\alpha$, so that

$$2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda = (a\cos\alpha - b\sin\alpha)/\tan\lambda.$$

Now multiplying by $\tan\lambda$ and collecting like terms gives

$$b\sin\alpha(1+\tan^2\lambda) = a\cos\alpha(1-\tan^2\lambda) - 2a\sin\alpha\tan\lambda.$$

Then using $1+\tan^2\lambda = \sec^2\lambda$ and then multiplying by $\cos^2\lambda$ gives us

$$\begin{aligned}
b\sin\alpha &= a\cos\alpha(\cos^2\lambda - \sin^2\lambda) - 2a\sin\alpha\sin\lambda\cos\lambda \\
&= a\cos\alpha\cos 2\lambda - a\sin\alpha\sin 2\lambda \\
&= a\cos(\alpha + 2\lambda),
\end{aligned}$$

as we wanted.

Marks

This markscheme follows the two approaches given using this method. If there is a significantly different attempt which is not fully correct, please discuss with PE. If it is not clear that the candidate is intending to use the Three Forces Theorem, and they do not get very far, please use the first mark scheme. (So no credit for working out angle of friction, etc., unless they have a clue what they're going to do with the information.)

M1: Determining angle of resultant reaction force to normal

A1 cao: Stating or deducing that this angle is λ

M1: Correctly stating or using the Three Forces Theorem

M1: Indicating all (combined) forces on diagram (condone at most one missing/extra force or one other error) with all forces passing through a single point

A1: Correct diagram with angles indicated (or implied)

M1: Calculating $PB = OB - OP$

M1: Reasonable attempt to determine PX from OA and $OP\tan\lambda$ (even if incorrect conclusion)

A1: Correct equation for PX resulting from this (does not need to be simplified)

M1: Combining results to get an equation $OB = OP + PB$ or $XP = PB/\tan\lambda$ with lengths appropriately substituted

M1: Simplifying resulting equation by expanding brackets and collecting terms or (in second approach) multiplying by $\tan\lambda$

M1: Rearranging to collect terms in a and b, and multiplying through by $\cos^2\lambda$

M1: Applying double angle formulæ to get $\cos 2\lambda$ and $\sin 2\lambda$

M1: Applying compound angle formula to get desired result

A1 cso: Correct solution reaching stated result

[Total for this part: 14 marks]

Show also that if the lamina is square, then $\lambda = \frac{\pi}{4} - \alpha$.

We have $a = b$ as the lamina is square, so that our previous equation becomes

$$a \sin \alpha = a \cos(2\lambda + \alpha).$$

Dividing by a gives

$$\sin \alpha = \cos(2\lambda + \alpha).$$

Now we can use the identity $\sin \alpha = \cos(\frac{\pi}{2} - \alpha)$, so that

$$\frac{\pi}{2} - \alpha = 2\lambda + \alpha.$$

(Being very careful, we should check that we can take inverse cosines of both sides to deduce this equality. This will be the case if both $0 < \frac{\pi}{2} - \alpha < \frac{\pi}{2}$ and $0 < 2\lambda + \alpha < \frac{\pi}{2}$. But $0 < \alpha < \frac{\pi}{2}$ so the first inequality is clearly true. For the second inequality, we have $0 < \lambda < \frac{\pi}{2}$ so that $0 < 2\lambda + \alpha < \frac{3\pi}{2}$. But since the cosine of this is positive (being $\sin \alpha$), it must lie in the range $0 < 2\lambda + \alpha < \frac{\pi}{2}$ as required.)

Subtracting α and dividing by 2 now gives our desired result:

$$\frac{\pi}{4} - \alpha = \lambda.$$

Marks

M1: Using $a = b$ to deduce $\sin \alpha = \cos(2\lambda + \alpha)$

M1: Using $\sin \alpha = \cos(\frac{\pi}{2} - \alpha)$

A1 cso: Deducing stated equality; does not require justification of the inverse cosine step

[Total for this part: 3 marks]

Question 10

A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \big(\mathrm{e}^t \cos t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t\big)\mathbf{j}.$$

Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement.

To find the velocity, $\mathbf{v}$, and acceleration, $\mathbf{a}$, we differentiate with respect to t (using the product rule).

We have

$$\begin{aligned}
\mathbf{r} &= \big(\mathrm{e}^t \cos t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t\big)\mathbf{j} \\
\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t &= \big(\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t + \mathrm{e}^t \cos t\big)\mathbf{j} \\
\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t &= \big((\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big) - (\mathrm{e}^t \sin t + \mathrm{e}^t \cos t))\mathbf{i} + \\
&\qquad \big((\mathrm{e}^t \sin t + \mathrm{e}^t \cos t) + (\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big)\mathbf{j} \\
&= (-2\mathrm{e}^t \sin t)\mathbf{i} + (2\mathrm{e}^t \cos t)\mathbf{j}.
\end{aligned}$$

We can rewrite these, if we wish, by taking out the common factors:

$$\begin{aligned}
\mathbf{r} &= \mathrm{e}^t\big((\cos t)\mathbf{i} + (\sin t)\mathbf{j}\big) \\
\mathbf{v} &= \mathrm{e}^t\big((\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j}\big) \\
\mathbf{a} &= 2\mathrm{e}^t\big((-\sin t)\mathbf{i} + (\cos t)\mathbf{j})\big).
\end{aligned}$$

From these, we can easily find the magnitudes of the displacement, velocity and acceleration:

$$\begin{aligned}
|\mathbf{r}| &= \mathrm{e}^t\sqrt{(\cos t)^2 + (\sin t)^2} = \mathrm{e}^t \\
|\mathbf{v}| &= \mathrm{e}^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} \\
&= \mathrm{e}^t\sqrt{2\cos^2 t + 2\sin^2 t} \\
&= \mathrm{e}^t\sqrt{2} \\
|\mathbf{a}| &= 2\mathrm{e}^t\sqrt{(-\sin t)^2 + (\cos t)^2} = 2\mathrm{e}^t.
\end{aligned}$$

We can now find the angles between these using $\mathbf{a}.\mathbf{b} = 2|\mathbf{a}||\mathbf{b}|\cos\theta$; firstly, for displacement and velocity we have

$$\begin{aligned}
\mathbf{r}.\mathbf{v} &= \mathrm{e}^{2t}\big(\cos t(\cos t - \sin t) + \sin t(\sin t + \cos t)\big) \\
&= \mathrm{e}^{2t}(\cos^2 t + \sin^2 t) \\
&= \mathrm{e}^{2t}, \qquad \text{while} \\
\mathbf{r}.\mathbf{v} &= \mathrm{e}^t.\mathrm{e}^t\sqrt{2}\cos\theta,
\end{aligned}$$

so that $\cos\theta = 1/\sqrt{2}$, so that $\theta = \frac{\pi}{4}$ as required.

Next, for displacement and acceleration we have

$$\mathbf{r}.\mathbf{a} = 2\mathrm{e}^{2t}\big(\cos t(-\sin t) + \sin t\cos t\big) = 0,$$

so they are perpendicular.

Geometric-trigonometric approach

There is another way to find the angles involved which does not use the scalar (dot) product.

Recall that the velocity is $\mathbf{v} = \mathrm{e}^t\big((\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j}\big)$. We can use the "$R\cos(\theta+\alpha)$" technique, thinking of $\cos t - \sin t$ as $1\cos t - 1\sin t$, so that

$$\begin{aligned}
\cos t - \sin t &= \sqrt{2}\big(\tfrac{1}{\sqrt{2}}\cos t - \tfrac{1}{\sqrt{2}}\sin t\big)\\
&= \sqrt{2}(\cos t\cos\tfrac{\pi}{4} - \sin t\sin\tfrac{\pi}{4})\\
&= \sqrt{2}\cos(t+\tfrac{\pi}{4})\\
\sin t + \cos t &= \sqrt{2}\big(\tfrac{1}{\sqrt{2}}\sin t + \tfrac{1}{\sqrt{2}}\cos t\big)\\
&= \sqrt{2}(\sin t\cos\tfrac{\pi}{4} + \cos t\sin\tfrac{\pi}{4})\\
&= \sqrt{2}\sin(t+\tfrac{\pi}{4})
\end{aligned}$$

Thus $\mathbf{v} = \sqrt{2}\mathrm{e}^t\big(\cos(t+\frac{\pi}{4})\mathbf{i} + \sin(t+\frac{\pi}{4})\mathbf{j}\big)$, so $\mathbf{v}$ is at an angle of $\frac{\pi}{4}$ with $\mathbf{r}$.

Likewise, $\mathbf{a}$ makes an angle of $\frac{\pi}{4}$ with $\mathbf{v}$, and so an angle of $\frac{\pi}{2}$ with $\mathbf{r}$.

Marks

M1: Use of product rule in either $\mathbf{i}$ *or* $\mathbf{j}$ *component (this implies first M1 mark) to determine* $\mathbf{v}$ *from* $\mathbf{r}$

A1 cao: Correct $\mathbf{v}$

M1: Reasonable attempt at differentiating $\mathbf{v}$ *to find* $\mathbf{a}$

A1 cao: Correct $\mathbf{a}$ *with simplified expressions; alternatively, if correct but unsimplified, yet this is successfully used to show that* $\mathbf{a}$ *is perpendicular to* $\mathbf{r}$*, then award this mark*

M1: Correct method for finding magnitudes of $\mathbf{r}$ *and* $\mathbf{v}$

A1: Correct $|\mathbf{r}|$ *and* $|\mathbf{v}|$ *(unsimplified)*

A1 ft: Correct $|\mathbf{r}|$ *and* $|\mathbf{v}|$ *simplified*

M1: Using $\mathbf{r}.\mathbf{v}$ *to find the angle between them*

A1: $\mathbf{r}.\mathbf{v} = \mathrm{e}^{2t}$

A1 cso: Showing angle is $\frac{\pi}{4}$

M1: Finding $\mathbf{r}.\mathbf{a}$

A1 cso: Deducing $\mathbf{r}$ *and* $\mathbf{a}$ *are perpendicular*

[Total for this part: 12 marks]

Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

One way of thinking about the path of the particle is that its displacement at time t is given by $\mathbf{r} = e^t\big((\cos t)\mathbf{i} + (\sin t)\mathbf{j}\big)$, so that it is at distance e^t from the origin and at an angle of t (in radians) to the x-axis (as $\big((\cos t)\mathbf{i}+(\sin t)\mathbf{j}\big)$ is a unit vector in this direction). Thus its distance at time $t = 0$ is $e^0 = 1$, and when it has gone a half circle, its distance is e^π, which is approximately $e^3 \approx 20$. So the particle moves away from the origin *very* quickly!

Another thing to bear in mind is that its velocity is always at an angle of $\frac{\pi}{4}$ to its displacement. Since it is moving away from the origin, its velocity is directed away from the origin, so initially it is moving at an angle of $\frac{\pi}{4}$ above the positive x-axis.

As we sketch the path, we also indicate the directions of the velocities at the times $t = 0$, $t = \frac{\pi}{2}$ and $t = \pi$.

Marks

B1: Either correct indication of points at $t = 0$, $t = \frac{\pi}{2}$ and $t = \pi$ or correct indication of one of these with correct direction of trajectory, or unlabelled points with general direction of trajectory correct at two or more of these points

B1: Correct indication of these three points and correct direction at them (if close to this, please discuss marks with PE)

[Total for this part: 2 marks]

> *A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.*

We write $\mathbf{r}_P = \mathbf{r}$ for the position vector of P and $\mathbf{r}_Q$ for the position vector of Q. We therefore have

$$\begin{aligned}
\mathbf{r}_P &= \big(e^t \cos t\big)\mathbf{i} + \big(e^t \sin t\big)\mathbf{j} \\
\mathbf{r}_Q &= \big(e^{t-T} \cos(t-T)\big)\mathbf{i} + \big(e^{t-T} \sin(t-T)\big)\mathbf{j},
\end{aligned}$$

and so we can calculate $|\mathbf{r}_P - \mathbf{r}_Q|^2$:

$$\begin{aligned}
|\mathbf{r}_P - \mathbf{r}_Q|^2 &= \big(e^t \cos t - e^{t-T}\cos(t-T)\big)^2 + \big(e^t \sin t - e^{t-T}\sin(t-T)\big)^2 \\
&= \big(e^{2t}\cos^2 t - 2e^t e^{t-T}\cos t\cos(t-T) + 2e^{2(t-T)}\cos^2(t-T)\big)+ \\
&\qquad \big(e^{2t}\sin^2 t - 2e^t e^{t-T}\sin t\sin(t-T) + e^{2(t-T)}\sin^2(t-T)\big) \\
&= e^{2t} - 2e^{2t-T}\big((\cos t\cos(t-T) + \sin t\sin(t-T)\big) + e^{2(t-T)} \\
&= e^{2t} - 2e^{2t-T}\cos(t-(t-T)) + e^{2t-2T} \\
&= e^{2t}\big(1 - 2e^{-T}\cos T + e^{-2T}\big),
\end{aligned}$$

so that

$$|r_P - r_Q| = e^t\sqrt{1 - 2e^{-T}\cos T + e^{-2T}},$$

which is clearly proportional to e^t, as required, since T is a constant.

Marks

B1: Correct position vector of Q

M1: Correct method for calculating $|\mathbf{r}_P - \mathbf{r}_Q|$ or $|\mathbf{r}_P - \mathbf{r}_Q|^2$

A1: Correct $|\mathbf{r}_P - \mathbf{r}_Q|^2$ (unsimplified OK for this mark)

M1: Simplification using $\cos^2 t + \sin^2 = 1$ and compound angle formula

A1: Determination of $|\mathbf{r}_P - \mathbf{r}_Q|^2$ as $e^{2t}f(T)$

A1 cso: Deducing stated conclusion

[Total for this part: 6 marks]

Question 11

Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions. Find the speeds of the particles after they collide for the first time and show that they will both change direction if $2em > M - m$.

This begins as a standard collision of particles question. ***ALWAYS*** *draw a diagram for collisions questions; you will do yourself (and the examiner) no favours if you try to keep all of the directions in your head, and you are very likely to make a mistake. My recommendation is to always have all of the velocity arrows pointing in the same direction. In this way, there is no possibility of messing up the Law of Restitution; it* **always** *reads $v_1 - v_2 = e(u_2 - u_1)$ or $\dfrac{v_1 - v_2}{u_2 - u_1} = e$, and you only have to be careful with the signs of the given velocities; the algebra will then keep track of the directions of the unknown velocities for you.*

Then Conservation of Momentum gives

$$Mu_1 + mu_2 = Mv_1 + mv_2$$

and Newton's Law of Restitution gives

$$v_2 - v_1 = e(u_1 - u_2).$$

Substituting $u_1 = u$ and $u_2 = -u$ gives

$$Mv_1 + mv_2 = (M - m)u \qquad (1)$$
$$v_2 - v_1 = 2eu. \qquad (2)$$

Then solving these equations (by (1) $- m \times$ (2) and (1) $+ M \times$ (2)) gives

$$v_1 = \frac{(M - m - 2em)u}{M + m} \qquad (3)$$
$$v_2 = \frac{(M - m + 2eM)u}{M + m}. \qquad (4)$$

The speeds are then (technically) the absolute values of these, but we will stick with these formulæ as they are what are needed later.

Now, the particles both change directions if v_1 and v_2 have the opposite signs from u_1 and u_2, respectively, so $v_1 < 0$ and $v_2 > 0$. Thus we need

$$M - m - 2em < 0 \quad \text{and} \qquad (5)$$
$$M - m + 2eM > 0. \qquad (6)$$

But (6) is always true, as $M > m$, so we only need $M - m < 2em$ from (5).

> ***Marks***
> *B1: Correct Conservation of Momentum (CoM) equation*
> *M1: Using Law of Restitution (LoR) correctly in context (condone at most one sign error for this mark)*
> *A1 cao: Correct resulting equation with u substituted appropriately*
> *M1: Solving equations simultaneously*
> *A1 cao: Correct v_1*
> *A1 cao: Correct v_2*
> *M1: Conditions on v_1 for particle to change direction*
> *M1: Conditions on v_2 for particle to change direction*
> *A1 cso: Justification that $v_2 > 0$*
> *A1 cso: Justification that $v_1 < 0$ precisely when $2em > M - m$ (answer given)*
> *[Total for this part: 10 marks]*

After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why

$$mv - MV = u(M - m),$$

and find v and V in terms of m, M, e, u and n.

The fact that the particles both change their directions of motion at each collision means that if they have velocities v_1 and v_2 after some collision, they will have velocities $-v_1$ and $-v_2$ before the next collision. This is because they are moving around a circular track, and therefore next meet on the opposite site, and hence are each moving in the opposite direction from the one they were moving in. (We do not concern ourselves with precisely where on the track they meet, and we are thinking of our velocities as one-dimensional directed speeds.)

Therefore, $mv_m + Mv_M$ is constant in value after each collision, where v_m is the velocity of the particle of mass m, and v_M that of the particle of mass M, but it reverses in sign before the next collision. So after the first collision, it it $Mu - mu$ to the right (in our above sketch), and hence after an even number of further collisions, it will still be $Mv_M + mv_m = Mu - mu$ to the right. But after an even number of further collisions, the particle of mass M is moving to the left, so $v_M = -V$, $v_m = v$. Thus

$$mv - MV = (M - m)u.$$

Also, since there are a total of $2n+1$ collisions, we have, by $2n+1$ applications of Newton's Law of Restitution,

$$V + v = e^{2n+1}(u + u).$$

Solving these two equations simultaneously as before then yields

$$V = \frac{(2me^{2n+1} - M + m)u}{M+m}$$
$$v = \frac{(2Me^{2n+1} + M - m)u}{M+m}.$$

Marks

B1: Explaining that if they separate after a collision with speeds v_1 and v_2, then they approach each other at the next collision with these same speeds (a reasonable explanation is adequate for this mark)

M1: Momentum is conserved at each collision ...

M1 dep: ... and reversed between consecutive collisions ...

A1 cso: ... so is $u(M-m)$ after further $2n$ collisions

If a candidate simply says that the momentum is the same at every collision and conclude that the momentum is $u(M-m)$ after further $2n$ collisions, only award B0 M1 M0 A0; in this case, though, they may still gain the final A1 mark below

If they short-circuit the argument (one candidate wrote something like "Odd number of collisions so particles directed in opposite direction from start, so by CoM, $u(M-m) = mv - MV$"), award B0 M1 M0 A1. If you think this is overly generous in a particular case, please discuss with PE

B1: Brief justification (picture implication by earlier argument is adequate) that momentum is $mv - MV$ after collisions

M1: Application of Newton's LoR $2n$ or $2n+1$ times

A1: Correct formula for $V + v$ from LoR

M1: Solving equations simultaneously

A1 cao: Correct v

A1 cao: Correct V

[Total for this part: 10 marks]

Question 12

A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ *for this case, and show that*

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} P(X \geqslant n).$$

For the definition of $\mathrm{E}(X)$, we simply plug the allowable values of X into the definition of $\mathrm{E}(X)$ for discrete random variables, to get

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} n\,\mathrm{P}(X = n).$$

Now, we can think of $n, \mathrm{P}(X = n)$ as the sum of n copies of $\mathrm{P}(X = n)$, so that we get

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} n\,\mathrm{P}(X = n) \\
&= 1.\mathrm{P}(X = 1) + 2.\mathrm{P}(X = 2) + 3.\mathrm{P}(X = 3) + 4.\mathrm{P}(X = 4) + \cdots \\
&= \mathrm{P}(X = 1) + \\
&\quad\ \mathrm{P}(X = 2) + \mathrm{P}(X = 2) + \\
&\quad\ \mathrm{P}(X = 3) + \mathrm{P}(X = 3) + \mathrm{P}(X = 3) + \\
&\quad\ \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \cdots
\end{aligned}$$

Adding each column now gives us something interesting: the first column is $\mathrm{P}(X = 1) + \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \cdots = \mathrm{P}(X \geqslant 1)$, the second column is $\mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \cdots = \mathrm{P}(X \geqslant 2)$, the third column is $\mathrm{P}(X = 3) + \mathrm{P}(X = 4) + \cdots = \mathrm{P}(X \geqslant 3)$, and so on. So we get

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{P}(X \geqslant 1) + \mathrm{P}(X \geqslant 2) + \mathrm{P}(X \geqslant 3) + \mathrm{P}(X \geqslant 4) + \cdots \\
&= \sum_{n=1}^{\infty} \mathrm{P}(X \geqslant n),
\end{aligned}$$

as we wanted.

An alternative, more formal, way of writing this proof is as follows, using what is sometimes called "summation algebra":

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} nP(X=n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n} \mathrm{P}(X=n) && \text{summing } n \text{ copies of a constant} \\
&= \sum_{1\leqslant m\leqslant n<\infty} \mathrm{P}(X=n) && \text{writing it as one big sum} \\
&= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \mathrm{P}(X=n) && \text{see below} \\
&= \sum_{m=1}^{\infty} \mathrm{P}(X\geqslant m),
\end{aligned}
$$

which is the sum we wanted. For the penultimate step, note the we are originally summing all pairs of values (m, n) where n is any positive integer and m lies between 1 and n, so we have $1 \leqslant m \leqslant n < \infty$, as written on the third line. This can also be thought of as summing over all pairs of values (m, n) where m is any positive integer (i.e., $1 \leqslant m < \infty$), and n is chosen so that $m \leqslant n < \infty$, that is, we are summing on n from m to ∞.

[One final technical note: we are allowed to reorder the terms of this *infinite* sum because all of the summands (the things we are adding) are non-negative. If some were positive and others were negative, we might get all sorts of weird things happening if we reordered the terms. An undergraduate course in Analysis will usually explore such questions.]

Marks

B1: Definition of $\mathrm{E}(X)$ *in this case (may be given with summation notation or as something like* $p_1 + 2p_2 + 3p_3 + \cdots$*; be generous with this mark as long as the candidate clearly knows what they mean (and sorry for the pun!); do* not *award for a generic* $\sum p_i x_i$*)*

M1: Expanding sum as $p_1 + (p_2 + p_2) + \cdots$

M1 dep: Adding at least one "column"

A1 cso: Reaching given conclusion

(It is unlikely that a candidate will use summation algebra in STEP I; please discuss any cases which attempt this but do not reach the conclusion with PE)

[Total for this part: 4 marks]

I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

We ask ourselves: what needs to happen to have $X \geqslant 4$? This means that we need to open at least 4 boxes to get both a daddy and a mummy penguin. In other words, we can't have had both a daddy and a mummy among the first three boxes, so they must have all had daddies or all had mummies. Therefore $\mathrm{P}(X \geqslant 4) = p^3 + q^3$.

This immediately generalises to give $\mathrm{P}(X \geqslant n) = p^{n-1} + q^{n-1}$, at least for $n \geqslant 3$. For $n = 1$, $\mathrm{P}(X \geqslant 1) = 1$, and for $n = 2$, $\mathrm{P}(X \geqslant 2) = 1 = p^1 + q^1$, as we argued above.

Therefore, we have

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} \mathrm{P}(X \geqslant n) \\
&= 1 + (p^1 + q^1) + (p^2 + q^2) + (p^3 + q^3) + \cdots \\
&= (1 + p + p^2 + p^3 + \cdots) + (1 + q + q^2 + q^3 + \cdots) - 1 \\
&= \frac{1}{1-p} + \frac{1}{1-q} - 1 \qquad \text{adding the geometric series} \\
&= \frac{1}{q} + \frac{1}{p} - 1 \qquad \text{as } p + q = 1 \\
&= \frac{p+q}{qp} - 1 \\
&= \frac{1}{pq} - 1 \qquad \text{again using } p + q = 1.
\end{aligned}$$

Marks

M1: Listing the possibilities giving rise to $X \geqslant 4$

M1 dep: Deducing probability of first three being either all mummies or all daddies

A1 cso: Deducing $\mathrm{P}(X \geqslant 4)$

M1: Generalising to $\mathrm{P}(X \geqslant n)$ (for $n \geqslant 2$, but award marks even without this condition explicitly stated)

A1: Deducing $\mathrm{P}(X \geqslant n)$ for $n \geqslant 2$ or $n \geqslant 3$ (but do not need to specify this condition to gain this mark); in the case that $n \geqslant 3$ is explicitly stated, will need to handle $n = 2$ case explicitly to gain the next-but-one M1 mark

M1: Stating conditions under which $\mathrm{P}(X \geqslant n) = p^{n-1} + q^{n-1}$ holds, or realising that need to handle $n = 1$ case separately

B1: $\mathrm{P}(X \geqslant 1) = 1$ *(this mark can only be awarded once a general formula for* $\mathrm{P}(X \geqslant n)$ *has been attempted)*

M1: Writing out sum for $\mathrm{E}(X)$ *explicitly*

M1: Breaking into sum of two infinite GPs, taking care of the -1 *term or having one of them beginning* $p + p^2 + \cdots$*, or equivalent*

A1: Summing the infinite GPs correctly

M1: Simplifying the fractions using $p + q = 1$ *at least once*

A1 cso: Reaching given result

An alternative for the first three marks:

M1: $\mathrm{P}(X \geqslant 4) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) - \mathrm{P}(X = 2) - \mathrm{P}(X = 3)$*, or without the* $\mathrm{P}(X = 0)$ *and* $\mathrm{P}(X = 1)$ *terms*

M1 dep: Determining $\mathrm{P}(X = 2)$ *and* $\mathrm{P}(X = 3)$*, getting at least one of them correct and the other close to correct*

A1 cso: Reaching given result

[Total for this part: 13 marks]

> *Hence show that* $\mathrm{E}(X) \geqslant 3$*.*

To show that $\mathrm{E}(X) \geqslant 3$, we simply need to show that $\frac{1}{pq} \geqslant 4$. But this is the same as showing that $pq \leqslant \frac{1}{4}$, by taking reciprocals.

Now, recall that $q = 1 - p$, so we need to show that $p(1-p) \leqslant \frac{1}{4}$. To do this, we rewrite the quadratic in p by completing the square:

$$p(1-p) = p - p^2 = \tfrac{1}{4} - (p - \tfrac{1}{2})^2.$$

Since $(p - \frac{1}{2})^2 \geqslant 0$ for all p (even outside the range $0 < p < 1$), we have $p(1-p) \leqslant \frac{1}{4}$, as required, with equality only when $p = q = \frac{1}{2}$.

This can also be proved using calculus, or using the AM–GM inequality, or by writing $1/pq = (p+q)^2/pq$ and then rearranging to get $(p-q)^2 \geqslant 0$.

Marks

This is just one way to do it. The mark scheme will need to be adapted to match the approach used.

M1: Rearranging to get $p - p^2 \leqslant \frac{1}{4}$

M1 dep: Either completing the square or differentiating w.r.t. p

A1: Correctly completing the square or correctly finding stationary point of function

A1 cso: Conclusion of proof; needs to be a solid argument to gain this final mark (in particular showing the correct direction of inequality), though condone use of $\Longrightarrow$ *instead of* $\Longleftrightarrow$ *in argument*

[Total for this part: 4 marks]

Question 13

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

Let X be the number of texts George receives in the first hour of the morning and Y be the number he receives in the second hour.

Then $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\lambda)$, with X and Y independent random variables.

We thus have

$$\begin{aligned}
\text{P(George waits between 1 and 2 hours for first text)} &= \text{P}(X = 0 \text{ and } Y > 0) \\
&= \text{P}(X = 0).\text{P}(Y > 0) \\
&= e^{-\lambda}.(1 - e^{-\lambda}) \\
&= p,
\end{aligned}$$

so that $e^{-\lambda} - e^{-2\lambda} = p$.

Multiplying this last equation by $e^{2\lambda}$ gives $e^{\lambda} - 1 = pe^{2\lambda}$; a straightforward rearrangement yields our desired equation.

(This equation can also be deduced by considering the *waiting time* until the first text; this is generally not studied until university, though.)

The solutions of the quadratic equation in e^{λ} are given by

$$e^{\lambda} = \frac{1 \pm \sqrt{1 - 4p}}{2p}.$$

But we are given that $4p < 1$, so that $1 - 4p > 0$ and there are real solutions. We need to show, though, that the two values of e^{λ} that we get are both greater than 1, so that the resulting values of λ itself are both greater than 0.

We have

$$\begin{aligned}
\frac{1 \pm \sqrt{1 - 4p}}{2p} > 1 &\iff 1 \pm \sqrt{1 - 4p} > 2p \\
&\iff \pm\sqrt{1 - 4p} > 2p - 1.
\end{aligned}$$

Now, since $4p < 1$, we have $2p < \frac{1}{2}$, so $2p - 1 < 0$, from which it follows that for the positive sign in the inequality, $\sqrt{1 - 4p} > 0 > 2p - 1$. It therefore only remains to show that $-\sqrt{1 - 4p} > 2p - 1$. But

$$\begin{aligned}
-\sqrt{1 - 4p} > 2p - 1 &\iff \sqrt{1 - 4p} < -(2p - 1) \\
&\iff 1 - 4p < (2p - 1)^2 \\
&\iff 1 - 4p < 4p^2 - 4p + 1,
\end{aligned}$$

which is clearly true as $4p^2 > 0$. (We were allowed to square between the first and second lines as both sides are positive.)

Thus the two solutions to our quadratic in e^λ are both greater than 1, so there are two positive values of λ which satisfy the equation.

Marks

M1: Considering two *relevant random variables, such as X = number in first hour, Y = number in second hour or U = number in first hour, V = number in first two hours*

M1 dep: Correct formula for p, such as $p = P(X = 0 \cap Y \geqslant 1)$ or $p = P(U = 0 \cap V \geqslant 1)$

M1 dep: If X and Y are used, noting that they are independent and hence determining probability as a product; if U and V are used, it is unlikely that the candidate will make much further meaningful progress without finally resorting to using X and Y, but discuss with the PE if they do make sensible progress down this route

A1 cso: Deducing stated quadratic

B1: Roots of this quadratic in e^λ

B1: Noting that $1 - 4p > 0$ so two real roots

M1: Rearranging inequality showing that roots are greater than 1 to get $\pm\sqrt{1-4p} > 2p - 1$

A1: Proving result for + case

M1: Squaring (correctly!) to handle − case

A1 cso: Deducing that there are two positive values of λ

An alternative method for the first four marks is to state explicitly *that T, the time until the first text has an exponential distribution with parameter λ, so that $P(1 < T < 2) = P(T < 2) - P(T < 1) = (1 - e^{-2\lambda}) - (1 - e^{-\lambda}) = e^{-\lambda} - e^{-2\lambda} = p$. Multiplying by $e^{2\lambda}$ then gives our result.*

This method can only be given credit if the candidate has clearly *shown that they know what they are doing, and then award as follows:*

M1: State that time till first text is exponential with parameter λ

A1: Correct $P(T < 1)$ or $P(T < 2)$

A1: Correct $P(1 < T < 2)$

A1 cso: Deducing stated equation

[Total for this part: 10 marks]

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables but with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Each phone behaves in the same way as George's phone above, so the two possible values of λ are those found above. That is, the values of e^{λ_1} and e^{λ_2} are the two roots of $pe^{2\lambda} - e^{\lambda} + 1 = 0$.

We know that the product of the roots of the equation $ax^2 + bx + c = 0$ is c/a,[1] so in our case, $e^{\lambda_1}e^{\lambda_2} = 1/p$, so that $e^{\lambda_1+\lambda_2} = 1/p$, giving

$$\lambda_1 + \lambda_2 = \ln(1/p) = -\ln p.$$

Marks

M1: Using result from first part to state that e^{λ_1} and e^{λ_2} are roots of quadratic

M1: Finding roots of quadratic and multiplying them or using product of roots result

A1: Product of roots is $1/p$

A1 cso: Finding $\lambda_1 + \lambda_2$

[Total for this part: 4 marks]

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Let X_1 be the number of texts she receives on the first phone during the first hour and Y_1 be the number of texts that she receives on the first phone during the second hour. Then X_1 and Y_1 are both distributed as $\text{Po}(\lambda_1)$, so

$$\text{P}(X_1 = 0) = e^{-\lambda_1}$$
$$\text{P}(Y_1 = 0) = e^{-\lambda_1}.$$

Now let X_2 and Y_2 be the corresponding random variables for the second phone, so we have

$$\text{P}(X_2 = 0) = e^{-\lambda_2}$$
$$\text{P}(Y_2 = 0) = e^{-\lambda_2}.$$

We must now consider the possible situations in which she receives her first text between 1 and 2 hours in the morning. She must receive no texts on either phone in the first hour, and at least one text on one of the phones in the second hour. We use the above result that $\lambda_1 + \lambda_2 = -\ln p$, so that $e^{-\lambda_1-\lambda_2} = p$.

[1]Why is this? If the roots of $ax^2 + bx + c = 0$ are α and β, then we can write the quadratic as $a(x-\alpha)(x-\beta) = a(x^2 - (\alpha+\beta)x + \alpha\beta)$, so that $c = a\alpha\beta$, or $\alpha\beta = c/a$. Likewise, $b = -a(\alpha+\beta)$ so that $\alpha + \beta = -b/a$.

Thus

$$
\begin{aligned}
&\mathrm{P}(\text{first text between 1 and 2 hours})\\
&\qquad = \mathrm{P}(X_1 = 0 \text{ and } X_2 = 0 \text{ and } Y_1 > 0 \text{ or } Y_2 > 0 \text{ or both})\\
&\qquad = \mathrm{P}(X_1 = 0).\mathrm{P}(X_2 = 0).\big(1 - \mathrm{P}(Y_1 = 0 \text{ and } Y_2 = 0)\big)\\
&\qquad = \mathrm{P}(X_1 = 0).\mathrm{P}(X_2 = 0).\big(1 - \mathrm{P}(Y_1 = 0).\mathrm{P}(Y_2 = 0)\big)\\
&\qquad = \mathrm{e}^{-\lambda_1}.\mathrm{e}^{-\lambda_2}.\big(1 - \mathrm{e}^{-\lambda_1}.\mathrm{e}^{-\lambda_2}\big)\\
&\qquad = \mathrm{e}^{-\lambda_1-\lambda_2}.(1 - \mathrm{e}^{-\lambda_1-\lambda_2})\\
&\qquad = p(1-p),
\end{aligned}
$$

and we are done.

Alternative approach

This approach uses a result which you may not have come across yet: the sum $X+Y$ of two independent Poisson random variables $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$ is itself a Poisson variable with $X+Y \sim \mathrm{Po}(\lambda+\mu)$.

Since the number of texts received on the two phones together is the sum of the number of texts received on each one, the total can be modelled by a Poisson random variable with mean $\Lambda = \lambda_1 + \lambda_2$ texts per hour.

Then the probability of waiting between 1 and 2 hours in the morning for the first text is given by q, where

$$q\mathrm{e}^{2\Lambda} - \mathrm{e}^{\Lambda} + 1 = 0,$$

using the result from the very beginning of the question, replacing p with q and λ with Λ. Since $\Lambda = \lambda_1 + \lambda_2 = \ln(1/p)$ from above, $\mathrm{e}^{\Lambda} = 1/p$.

Therefore

$$
\begin{aligned}
q &= \frac{\mathrm{e}^{\Lambda} - 1}{\mathrm{e}^{2\Lambda}}\\
&= \frac{1/p - 1}{(1/p)^2}\\
&= p^2(1/p - 1)\\
&= p(1-p).
\end{aligned}
$$

Marks

M1: Using two relevant and correct independent random variables for each of the two phones, or an equivalent method

A1: Correct probabilities for chosen (simple) events in terms of λ_i or p

M1: Combining the events correctly to find an event which is equivalent to "first text between 1 and 2 hours"

M1 dep: Splitting event using independence into simpler events

A1: Probability in terms of λ_i (possibly with p appearing somewhere)

A1 cao: Probability in terms of p

[Total for this part: 6 marks]